MST 204.2

The Open University
Mathematics/Science/Technology
An Inter-faculty Second Level Course
MST204 Mathematical Models and Methods

mathematical models and methods

# Unit 2
# Differential equations I

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# Differential equations I

Prepared for the Course Team
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# Introduction

Mathematical modelling of reality usually results in an equation or a set of equations. These equations can have a variety of forms, and the nature of the solutions to the equations depends on the nature of the equations.

You should already be familiar with ordinary *algebraic* equations, whose solutions are *numbers*. For instance, the algebraic equation

$$x^2 - 2 = 0$$

has a solution

$$x = \sqrt{2} = 1.414\ldots .$$

In *Unit 1*, you studied equations of a different type called *recurrence relations*. These are equations whose solutions are *sequences*. For example, we saw that the sequence whose $n$th term $u_n$ is defined by

$$u_n = 2^n$$

is a solution of the recurrence relation

$$u_{r+1} = 2u_r \qquad (r = 1, 2, 3, \ldots)$$

The equations we study in this unit are of a different kind again; they are called **differential equations**. Such equations are widely used in mathematical models. You will see examples in many units of this course, starting with *Units 3* and *4*. An example of a differential equation is

$$\frac{dy}{dx} = x + y \tag{1}$$

You will see that it contains two variables $x$ and $y$. The problem posed by a differential equation is not to find numerical values such as $\sqrt{2}$ for the variables in it but to find a *function* relating these variables. One way of describing such a function is to give a formula expressing $y$ in terms of $x$, for example the formula

$$y = e^x - x - 1 \tag{2}$$

describes one such function. In fact the particular function described by Formula (2) happens to be a solution of Equation (1); for if $x$ and $y$ are related by Formula (2) then the left-hand side of Equation (1) becomes

$$\frac{dy}{dx} = \frac{d}{dx}(e^x - x - 1) = e^x - 1$$

and the right-hand side becomes

$$x + y = x + (e^x - x - 1) = e^x - 1.$$

That is, Formula (2) makes the two sides of Equation (1) equal and so the formula gives a solution of the differential equation.

By the end of this unit you will know how this solution was obtained, and you will be able to find all the other solutions of the differential equation as well.

There is an alternative notation for differential equations and their solutions. Instead of using two variables, $x$ and $y$ say, this alternative notation uses just one variable, $x$ say, together with a symbol for the function the differential equation informs us about. For example, if this function were denoted by $U$, then Equation (1) would take the form

This is the notation used in *M101* and *MS283*.

$$\frac{dU(x)}{dx} = x + U(x)$$

or

$$U'(x) = x + U(x)$$

and Formula (2) would be written

$$U(x) = e^x - x - 1$$

or

$$U : x \longmapsto e^x - x - 1.$$

In this course, however, following the usual practice of applied mathematicians, scientists and technologists we shall not normally introduce a symbol for the function.

To translate from one notation to the other, use the following 'dictionary'

| **this course** | **function notation** |
|---|---|
| $y$ | $U(x)$ |
| $\frac{dy}{dx}$ | $\frac{dU(x)}{dx}$ or $U'(x)$ |

## Study guide

There are two main methods for solving differential equations: analytical methods which provide the solution function as a formula like Formula (2) above, and numerical methods which provide the solution function in the form of a table. In addition, there are graphical methods which provide information about the graph of the solution function.

This unit consists of two parts which can be studied independently. One part, consisting of Sections 1, 2 and 6, is concerned with numerical and graphical methods. It begins with a television programme on some basic ideas about differential equations, including a discussion of some graphical techniques; it then proceeds, in Section 2, to describe a simple numerical method. Section 6 describes the work in your first computer terminal visit, part of which is designed to help you learn to use the numerical method described in Section 2. Be sure to plan this terminal visit in good time: there may be a deadline after which you cannot use the terminal. For up-to-date information about this consult the *Course Guide* and stop press notices.

The other part of the unit, consisting of Sections 3 and 4, describes some analytical methods for solving differential equations. You can do this part of the unit either before or after the television programme and computer terminal visit. There is a tape commentary which forms part of Section 4.

Section 5 contains exercises to enable you to revise and consolidate what you have learned from the rest of the unit. There is no essentially new material in this section.

The order in which you study the sections of this unit depends on how the television broadcast schedule fits in with your study schedule and also on when you can arrange your visit to the computer terminal. If you are ready to start reading the unit before the television programme is due to be broadcast, begin by studying Sections 3 and 4, returning to Sections 1 and 2 after watching the programme. Otherwise start by watching the television programme and then study the unit in the order it is written. Whichever part of the unit you read first make sure you read through the preliminary work at the beginning of Section 1 *before* watching the television programme. Before going on your computer terminal visit you must have studied Section 6. This can be done any time after studying Sections 1 and 2.

# 1 Direction fields (Television Section)

## 1.1 Preliminary work

This section describes an approach to differential equations starting from the concept of *direction field*. This concept is explained in the television programme, but the following exercise will help you to understand the basic ideas more easily.

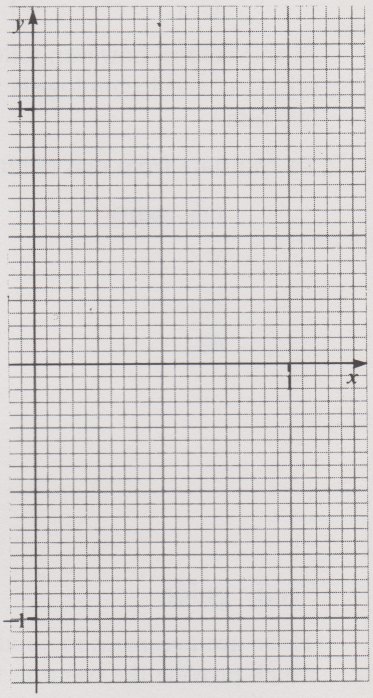


Figure 1

**Exercise 1**

On the graph paper provided in Figure 1 draw short line segments
(i) through $(0, 0)$ with slope 1
(ii) through $(\frac{1}{2}, \frac{1}{2})$ with slope 0
(iii) through $(\frac{1}{2}, -\frac{1}{2})$ with slope $-\frac{2}{3}$
[*Solution on p. 43*]

The television programme looks at the practical problem of finding curved mirrors which focus a parallel beam of light towards a particular point (see Figure 4). Using this problem as an example the programme shows how any direction field leads, in a natural way, to a set of curves (called *trajectories*). The method used to find these curves is illustrated by various examples. Exercises 2 and 3 ask you to check some of the results used in these examples.

**Exercise 2**

(i) Given that $y = \frac{1}{2}x^2 + 5$, check that

$$\frac{dy}{dx} = x.$$

(ii) Given that $y = \frac{1}{2}x^2 + C$, where $C$ is any constant, check that

$$\frac{dy}{dx} = x.$$

(iii) Given that $y = Ce^x - x - 1$, where $C$ is any constant, check that

$$\frac{dy}{dx} = x + y.$$

[*Solution on p. 43*]

**Exercise 3**

(i) Given that $y = (R^2 - x^2)^{1/2}$, where $R$ is any constant, check that

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$y = (R^2 - x^2)^{1/2}$ is the equation of a semi-circle of radius $R$.

To keep the calculation under control I suggest the following systematic procedure.

(a) Calculate $\frac{dy}{dx}$ from the given formula for $y$.

(b) Calculate $-\frac{x}{y}$ from the given formula for $y$.

(c) Check that the results of (a) and (b) are the same.

(ii) Given that $y = \frac{x^2}{2C} - \frac{C}{2}$, with $C > 0$, use a similar systematic procedure to check that

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

[*Solution on p. 43*]

During the programme we shall need to find the orientation of a reflecting surface which will deflect a ray of light towards a particular point. To get a formula for this orientation we need the law of reflection (see Figure 2). This is an empirical law which tells us that the normal (i.e. the perpendicular) to the reflector bisects the angle between the incident and reflected rays at the point of reflection. That is to say, the angles $\alpha$ and $\beta$ in Figure 2 are equal. Since $\alpha + \theta$ and $\beta + \phi$ are also equal (both being right angles) it follows that

$$\theta = \phi.$$

This relation is used as part of a geometrical derivation in the programme.

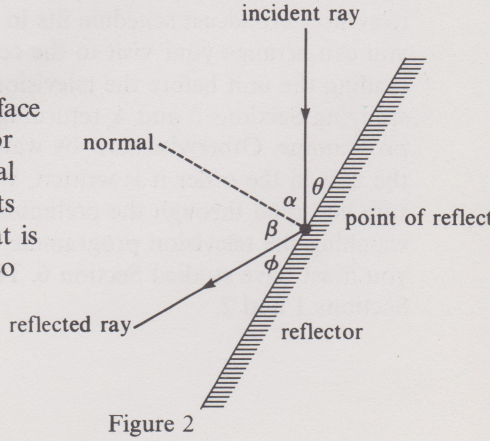


Figure 2

In order to follow this derivation you also need to know the following two geometrical theorems (for how they are used, see Figure 9).

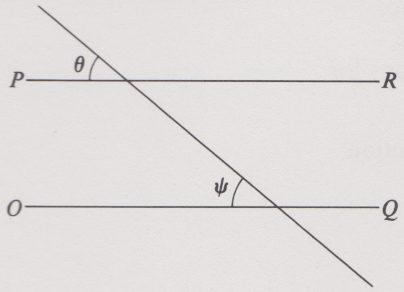



Figure 3(a). If the lines $OQ$ and $PR$ are parallel, then the angles $\psi$ and $\theta$ are equal.

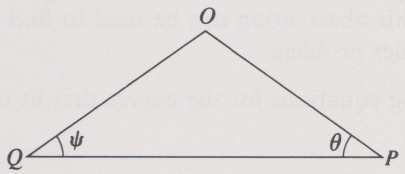
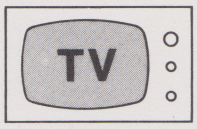



Figure 3(b). If the angles $\psi$ and $\theta$ are equal, then the lengths of sides $OQ$ and $OP$ are equal. A triangle such as $OQP$ with two equal sides is said to be **isosceles**.

*Now view the television programme 'Direction fields and families of curves'.*




TV2

## 1.2 'Direction fields and families of curves'

In this programme we look at the problem of focusing a parallel beam of light to one point called the focus: we want to formulate a mathematical equation that describes a curved reflector which will do this (see Figure 4).

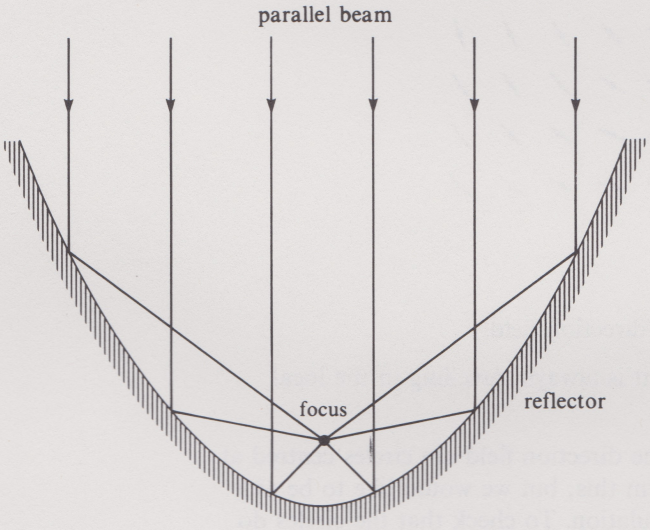



Figure 4

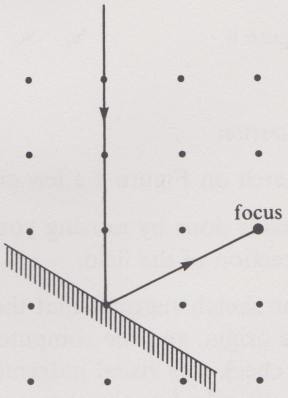



Figure 5

To analyse the problem, we look at one ray of light selected from the beam and show that if a small reflecting surface is placed at a point, then there is only one orientation of this surface for which the ray is deflected towards the focus (see Figure 5).

By considering other points in the plane, we obtain a whole set of directions. Every point in the plane (except for the focus) has an associated direction which is unique and we can determine it by experiment. Such an association of each point with a unique direction is called a **direction field** (see Figure 6).

You can also think of a direction field as a function which maps points in the plane to slopes.

Figure 6

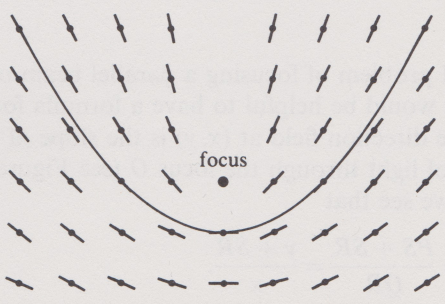



Figure 7

At the beginning of the programme, we showed three curves that had the required property of focusing a parallel beam of light onto one point. By superimposing one of these curves on the direction field (see Figure 7), we immediately notice that the slope of the curve at any point coincides with the slope given by the direction field. To illustrate how this observation can be used to find equations for the curves we first consider a simpler problem.

We consider the problem of finding equations for the curves that fit the direction field given by the formula

$$\text{slope at } (x, y) = -\frac{x}{y}.$$

A diagram of this field is shown in Figure 8.

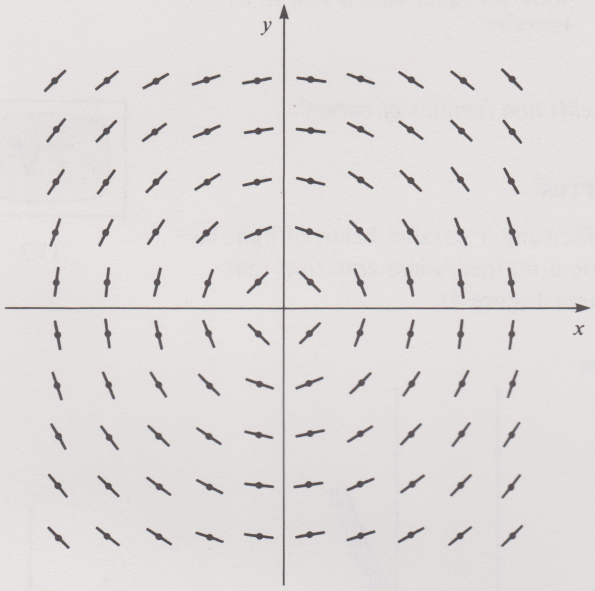



Figure 8

*Question*

Sketch on Figure 8 a few curves that fit the direction field.

This is done by moving your pencil so that it is always travelling in the local direction of the field.

The sketch suggests that the curves fitting the direction field are circles centred at the origin, and the computer graphics confirm this; but we would like to be able to check this visual judgement by exact calculation. To check that the circles do fit, we calculate the slopes of the tangents to the circles. The equation of a circle of radius $R$, centred at the origin, is $x^2 + y^2 = R^2$ and so expressing $y$ as a function of $x$ we obtain (for the part of the circle where $y > 0$)

$$y = (R^2 - x^2)^{1/2}.$$

Differentiating, we obtain for the slope of the tangent at $(x, y)$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{-2x}{(R^2 - x^2)^{1/2}}\right) = -\frac{x}{y}.$$

see Exercise 3(i).

So we have checked that the family of circles do fit the direction field whose slope at $(x, y)$ is $-\dfrac{x}{y}$.

Now we return to our original problem of focusing a parallel beam of light. The example above suggests that it would be helpful to have a formula for the direction field. The slope of the direction field at $(x, y)$ is the slope of the mirror which reflects the vertical ray of light through the focus $O$ (see Figure 9). By considering the triangle $PRQ$ we see that

$$\text{slope at } (x, y) = \frac{PR}{QR} = \frac{PS + SR}{QR} = \frac{y + SR}{x}.$$

We are using the convention that, for example, $PR$ is the *length* of the line from $P$ to $R$.

Since $OQRS$ is a rectangle this becomes

$$\text{slope at } (x, y) = \frac{y + OQ}{x}. \tag{1}$$

Now consider the angles. Since $OQ$ and $PS$ are parallel we have (see the theorem at the end of Subsection 1.1)

$$\psi = \theta.$$

Also, by the law of reflection we have

$$\theta = \phi.$$

It follows that

$$\psi = \phi.$$

Thus two angles in triangle $OPQ$ are equal, and so the corresponding sides are equal:

$$OQ = OP$$
$$= \sqrt{x^2 + y^2} \qquad \text{(by Pythagoras).}$$

Substituting this into Formula (1) we see that the formula for the direction field is

$$\text{slope at } (x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

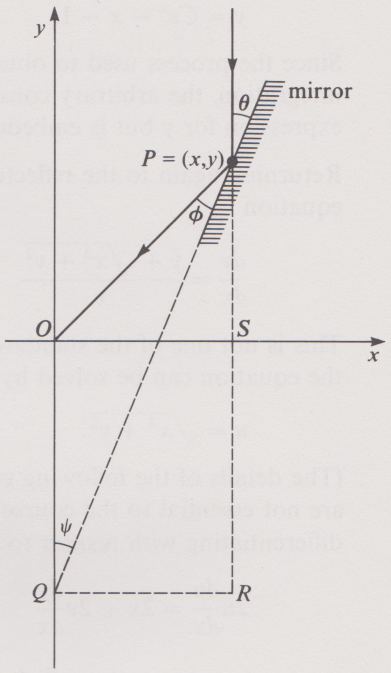


Figure 9

What condition must be satisfied by a curve that fits this direction field? Suppose the equation of the curve is

$$y = f(x)$$

where $f$ is a function we do not know yet. The slope of a tangent to the curve is $dy/dx$, the derivative of $f(x)$. If the curve is to fit the direction field this slope must match the slope of the direction field, that is, we must have

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}. \tag{2}$$

An equation like this, connecting the derivative $\dfrac{dy}{dx}$ of some unknown function to the variables $x$ and $y$ which are related by it, is called a (**first order**) **differential equation**.

There are many different curves that fit the direction field, and each one of them corresponds to a different function whose derivative satisfies the differential equation. These functions are called **solutions** of the differential equation.

Our original reflector problem has now been reduced to the problem of finding solutions of Differential Equation (2). At first sight, finding the solutions of a differential equation may seem like a formidable task. Fortunately it is often possible to find a method which gives all the solutions in terms of a single formula. To see how this can be done, we again digress from the reflector problem and consider a simpler differential equation

$$\frac{dy}{dx} = x.$$

The solutions of this differential equation can be found by integrating both sides with respect to $x$ to give the formula,

$$y = \tfrac{1}{2}x^2 + C$$

where $C$ is an arbitrary constant of integration. Each numerical value for $C$ gives a different function relating $y$ to $x$, and as you checked in Exercise 2 each of these functions satisfies (i.e. is a solution of) the differential equation. Each of these solutions corresponds to a different curve fitting the direction field whose slope at $(x, y)$ is $x$ (see Figure 10).

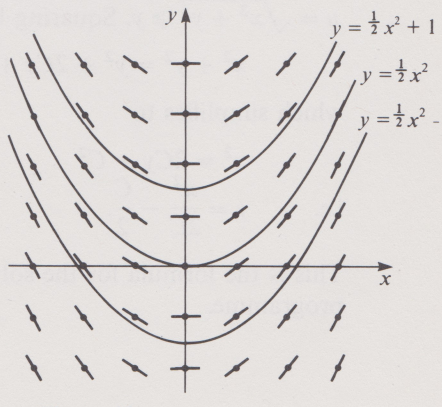


Figure 10

A more complicated example is given by the differential equation

$$\frac{dy}{dx} = x + y.$$

The solutions of this differential equation can be found by the method described in Section 4 of this unit. They are described by the formula

$$y = Ce^x - x - 1.$$

Since the process used to obtain this formula is more complicated than a direct integration, the arbitrary constant $C$ is this time not just added on to the expression for $y$ but is embedded in it in a more complicated way.

Returning again to the reflector problem, we want to solve the differential equation

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}.$$

This is not one of the standard types considered in this unit, but it happens that the equation can be solved by introducing a new variable $u$, defined by

$$u = \sqrt{x^2 + y^2}. \tag{3}$$

(The details of the following calculation are not in the television programme and are not essential to the course). Squaring both sides of Equation (3) and then differentiating with respect to $x$ gives

$$\begin{aligned} 2u\frac{du}{dx} &= 2x + 2y\frac{dy}{dx} \\ &= 2x + 2y\left(\frac{y + \sqrt{x^2 + y^2}}{x}\right) \qquad \text{(from the differential equation)} \\ &= 2\left(\frac{x^2 + y^2 + y\sqrt{x^2 + y^2}}{x}\right) \\ &= 2\left(\frac{u^2 + yu}{x}\right) \qquad \text{(by definition of } u\text{)} \end{aligned}$$

so that

$$\begin{aligned} \frac{du}{dx} &= \frac{u + y}{x} \\ &= \frac{\sqrt{x^2 + y^2} + y}{x} \\ &= \frac{dy}{dx}. \qquad \text{(from the differential equation)} \end{aligned}$$

It follows by integrating both sides with respect to $x$ that

$$u = y + C$$

where $C$ is a constant of integration, which must be positive since $u = \sqrt{x^2 + y^2} \geqslant y$. Squaring both sides and then using the definition of $u$ gives

$$x^2 + y^2 = y^2 + 2Cy + C^2$$

which simplifies to

$$x^2 = 2Cy + C^2$$

i.e. $$y = \frac{x^2}{2C} - \frac{C}{2}.$$

This is the formula for the solution of the reflector problem used in the programme.

## Summary of Section 1

The main points made in the programme are these:

1. A **direction field** specifies a slope at each point in the $(x, y)$ plane.

2. A **first order differential equation** is an equation connecting the derivative $dy/dx$ of some unknown function to the variables $x$ and $y$ which are related by that function.

   If $y = f(x)$ is the equation of a curve that smoothly fits the direction field, then the function $f$ must satisfy the differential equation obtained by equating $dy/dx$ to the expression for the slope of the direction field. Any function (that is, any relation giving $y$ in terms of $x$) which satisfies the differential equation is called a **solution** of the differential equation. There is an infinite family of solutions, each one corresponding to a different curve that fits the direction field.

   These curves are often called **trajectories**.

   These functions are sometimes called **particular solutions**.

3. If an exact formula can be found for the solutions of the differential equation then this formula will contain an arbitrary constant of integration, called $C$ in this programme. Different numerical values for $C$ give different solutions and therefore label different curves fitting the direction field.

   This formula is called the **general solution** of the differential equation.

4. Whether or not it is possible to find a formula for the solution of a differential equation, the direction field is a useful source of information about its solution; in particular it can be used to sketch solution curves and so get an idea of their qualitative behaviour.

**Exercise 4**

Draw the direction field whose slope is 2 at every point in the plane. Can you deduce the family of curves associated with this field? Check your result by differentiation.
[*Solution on p. 43*]

**Exercise 5**

On p. 8 we checked that the upper half of the circles matched the direction field. Can you do the same for the lower half?
[*Solution on p. 43*]

**Exercise 6**

What would be the differential equations for the trajectories of the direction fields defined by

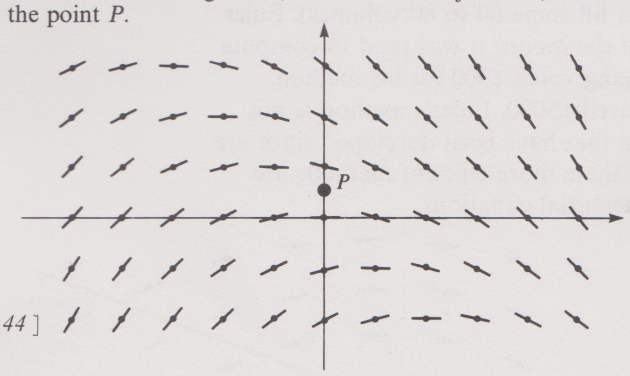
(i) Slope at $(x, y) = -x/y$
(ii) Exercise 4
(iii) The direction at any point $P$ (other than the origin) is the same as that of the line $OP$ where $O$ is the origin.

[*Solution on p. 43*]

**Exercise 7**

For the direction field shown in Figure 11, sketch a few trajectories, including the one that passes through the point $P$.

Figure 11




[*Solution on p. 44*]

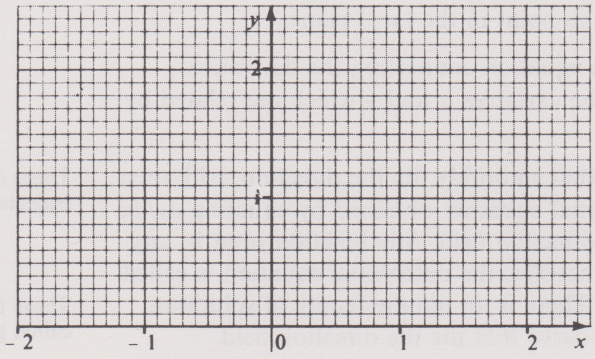
**Exercise 8**

Using the graph paper in Figure 12, draw some slopes belonging to the direction field defined by

slope at $(x, y) = 1 - y$

including the ones at the points $(0,0)$, $(0,\frac{1}{2})$, $(0,1)$, $(0,1\frac{1}{2})$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$. Hence sketch enough slopes to give a good idea of how the direction field behaves in the region shown.

Figure 12




[*Solution on p. 44*]

**Exercise 9**

Use your direction field from the preceding exercise to sketch some solution curves of the differential equation

$$\frac{dy}{dx} = 1 - y$$

including the ones passing through the points $(0,0)$, $(0,1)$ and $(0,2)$.
[*Solution on p. 44*]

# 2 A numerical method

## 2.1 A geometrical description of the method

So far the only *systematic* method we have for solving differential equations is to draw the direction field and sketch its trajectories. In the rest of this unit we develop some other methods for solving differential equations.

For certain types of differential equation there are exact methods of doing this which give the solution as a formula such as $y = e^x - x - 1$. Some of these methods are considered in Sections 3 and 4; but there are many differential equations for which these exact methods do not work and an approximate (numerical) method of solution must be used. Most numerical methods yield the solution not as a formula but as a table of values, similar to a table of logarithms.

In this section we shall study the simplest numerical method for differential equations. The method is known as **Euler's method**, after Leonhard Euler (pronounced Oiler), who lived from 1707 to 1783 and was one of the most prolific mathematicians of all time (his complete works fill some 60 to 80 volumes). Euler devised this method for computing the orbit of the moon; it was used to compute lunar tables for the British Admiralty, Euler being voted £300 for his method, while the person who did the calculations received £5000. Euler's method is not used much now but the more efficient methods that have been developed since are based on similar principles. Details of some of these more efficient methods are given in the unit on numerical solutions of differential equations.

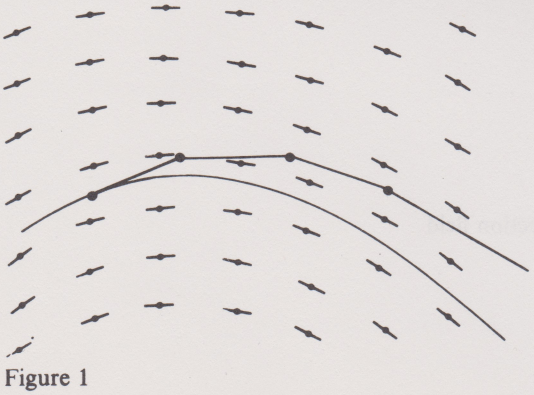


Figure 1

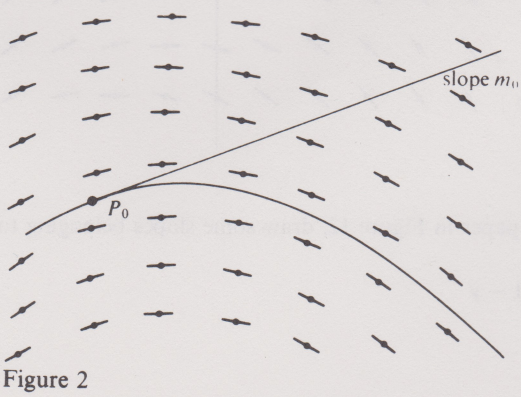



Figure 2

The idea behind Euler's method is to approximate the trajectories of the direction field associated with the differential equation by segments of straight lines joined end to end to form an open polygon as shown in Figure 1.

Before discussing how to construct this polygonal approximation we must first specify which of the infinite family of trajectories we wish to approximate. Let us suppose that the trajectory has been specified by saying that it passes through a certain point $P_0$ with co-ordinates $(x_0, y_0)$. Such a specification is called an **initial condition**; it provides a starting point from which the polygonal approximation can be built up step by step as follows. The first vertex of the approximating polygon is taken to be the point $P_0$ itself. The first of the line segments forming the polygon starts at $P_0$ and is taken to have the same slope as the direction field at $P_0$; let us call this slope $m_0$ (see Figure 2). This choice makes the first segment a tangent at $P_0$ to the trajectory we want, and gives it the right slope at $P_0$; but as we proceed along this line segment its slope will deviate more and more from the slope of the direction field. If this error is not to become too large, we must soon make a change of direction. Let us call the point at which we decide to make this change of direction $P_1$: it is the next vertex of our polygon (see Figure 3). At this point we begin the next line segment of the polygon, giving it the slope of the direction field at $P_1$, which we denote by $m_1$. This new segment is continued to some point $P_2$ (see Figure 4) at which we once again change direction, the next segment having slope $m_2$ equal to the slope of direction field at $P_2$. Continuing this process we obtain the open polygon shown in Figure 1.

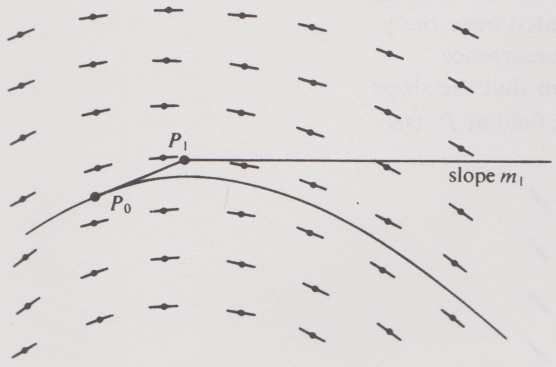



Figure 3

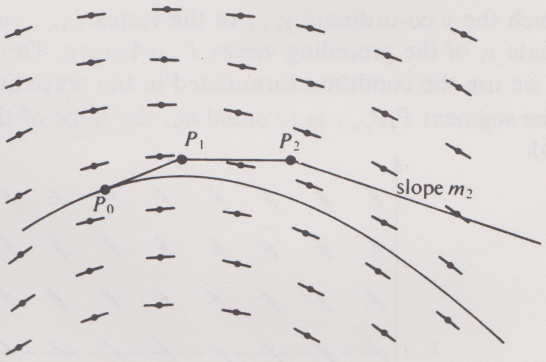



Figure 4

To follow the trajectory perfectly we would have to adjust the slope of our constructed figure continuously to that of the direction field, but in this approximation we adjust it only at the points $P_0, P_1, P_2, \ldots$ .

**Exercise 1**

The diagram below shows a direction field whose slope at the point $(x, y)$ is $x + y$. Using the graphical construction just described, draw on the diagram a polygonal approximation to the trajectory passing through the origin $P_0$. Choose the points $P_1, P_2, \ldots, P_5$ so that their $x$ co-ordinates are $0.2, 0.4, \ldots, 1.0$. There is no need to make any detailed calculations; a rough sketch is all that is required.

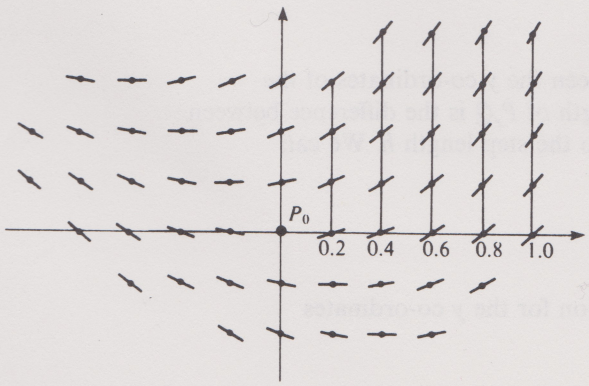



Figure 5

[*Solution on p. 44*]

## 2.2 A formula for Euler's method

If Euler's method were no more than a graphical construction, it would offer little advantage in comparison with sketching solution curves directly from the direction field, which is the method we used in Section 1. The point of Euler's method, however, is that it can, with advantage, be carried out numerically rather than graphically. Instead of a graph showing the polygonal approximation to the trajectory, the numerical approach yields a table listing the $x$ and $y$ co-ordinates of the vertices $P_0, P_1, \ldots$. By referring to this table we can read off the approximation for $y$ corresponding to the value of $x$ we are interested in.

In order to simplify the calculation of the $x$ co-ordinates we choose the vertices $P_0, P_1, \ldots$ so that their $x$ co-ordinates $x_0, x_1, \ldots$ are equally spaced. The spacing is called the **step length** and is denoted by $h$. For example, in Exercise 1 the $x$ co-ordinates were $0, 0.2, 0.4, \ldots$ and so the step length $h$ was 0.2. In general, suppose the trajectory we are trying to approximate has been specified by giving the co-ordinates $(x_0, y_0)$ of the point $P_0$. If the step length is $h$ then the $x$ co-ordinate of $P_1$ is given by $x_1 = x_0 + h$. By adding another step length $h$ we obtain the $x$ co-ordinate of $P_2$; $x_2 = x_0 + 2h$. Continuing in this way we obtain the general formula,

$$x_r = x_0 + rh$$

for the $x$ co-ordinate of the vertex $P_r$.

The $y$ co-ordinates of the vertices are calculated by setting up a recurrence relation from which the $y$ co-ordinate $y_{r+1}$ of the vertex $P_{r+1}$ can be calculated once the $y$ co-ordinate $y_r$ of the preceding vertex $P_r$ is known. To obtain this recurrence relation we use the condition formulated in the preceding subsection that the slope of the line segment $P_rP_{r+1}$ is to equal $m_r$, the slope of the direction field at $P_r$ (see Figure 6).

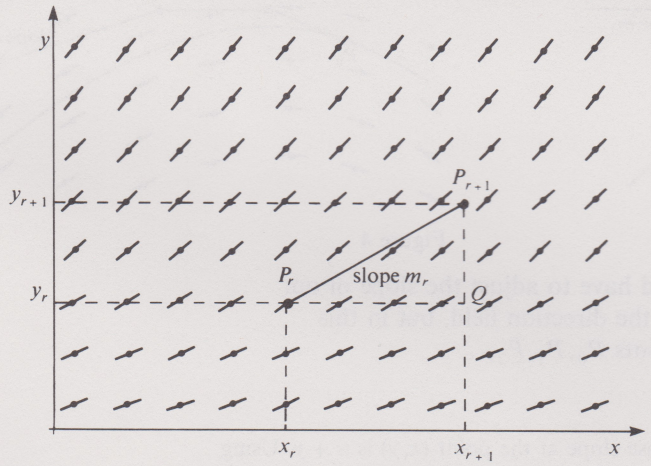



Figure 6

From the right-angled triangle shown in Figure 6 the condition for the line segment $P_rP_{r+1}$ to have slope $m_r$ is

$$\frac{\text{length of } P_{r+1}Q}{\text{length of } P_rQ} = m_r.$$

Now the length of $P_{r+1}Q$ is the difference between the $y$ co-ordinates of the vertices $P_{r+1}$ and $P_r$, that is $y_{r+1} - y_r$. The length of $P_rQ$ is the difference between the $x$ co-ordinates of $P_{r+1}$ and $P_r$; it is equal to the step length $h$. We can therefore write

$$\frac{y_{r+1} - y_r}{h} = m_r.$$

Solving for $y_{r+1}$ we obtain the recurrence relation for the $y$ co-ordinates

$$y_{r+1} = y_r + hm_r.$$

Since the $y$ co-ordinate of the point $P_0$ is known, the $y$ co-ordinates of the remaining vertices can be calculated using this recurrence relation. However, before we can do this we need a formula for the slope $m_r$ of the direction field at

the point $(x_r, y_r)$. This formula is provided by the differential equation we are trying to solve: for example if the differential equation is

$$\frac{dy}{dx} = x + y$$

then the formula for $m_r$ is

$$m_r = x_r + y_r.$$

For a general differential equation of the type considered in this unit, say

$$\frac{dy}{dx} = m(x, y)$$

where $m(x, y)$ can stand for any formula involving one or both of the variables $x$ and $y$, the corresponding formula for $m_r$ is

$$m_r = m(x_r, y_r).$$

To sum up we have:

**Procedure 2.2: Euler's method**

To apply Euler's method to the differential equation

$$\frac{dy}{dx} = m(x, y)$$

given the initial condition, $y = y_0$ when $x = x_0$, using step length $h$;

1. Use the co-ordinates $x_0$, $y_0$ as a starting point.
2. Calculate the next $y$ co-ordinate using the recurrence relation

$$y_{r+1} = y_r + hm_r$$

where $m_r = m(x_r, y_r)$.

3. Calculate the corresponding $x$ co-ordinate using the formula

$$x_r = x_0 + rh$$

(after increasing the value of $r$ by 1).

4. If more co-ordinates are required return to Step 2.

The recurrence relation for Euler's method can also be derived without reference to a graphical construction. By the definition of a derivative, the differential equation is equivalent to

$$\lim_{h \to 0} \frac{y(x + h) - y(x)}{h} = m(x, y).$$

Hence, for sufficiently small $h$, we have the approximation

$$\frac{y(x + h) - y(x)}{h} \simeq m(x, y).$$

This holds for all values of $x$, in particular for $x = x_r$; so we can replace $x$ by $x_r$ in the formula and rearrange to obtain

$$y(x_r + h) \simeq y(x_r) + hm(x_r, y(x_r)).$$

With the notation $y(x_r) = y_r$, $y(x_r + h) = y_{r+1}$ this is the same as the recurrence relation in the 'Euler's method' box.

**Example 1**

Let us apply Euler's method numerically to the problem treated graphically in Exercise 1. In that exercise we considered the direction field corresponding to the differential equation

$$\frac{dy}{dx} = x + y$$

and constructed a polygonal approximation to the trajectory passing through the origin. The step length was 0.2.

Since our polygonal approximation started at the origin, we take $P_0$ to be the origin. So the starting point for our calculation is

$$x_0 = 0,\ y_0 = 0, \qquad m_0 = x_0 + y_0 = 0.$$

According to Procedure 2.2, the next $y$ co-ordinate is

$$y_1 = y_0 + 0.2m_0 = 0$$

and the corresponding $x$ co-ordinate is

$$x_1 = x_0 + 1 \times 0.2 = 0.2$$

which tells us all we need to know about $P_1$:

$$x_1 = 0.2,\ y_1 = 0, \qquad m_1 = x_1 + y_1 = 0.2.$$

Next we obtain the co-ordinates of $P_2$:

$$y_2 = y_1 + 0.2m_1 = 0.04$$

and

$$x_2 = x_0 + 2 \times 0.2 = 0.4$$

so we have

$$x_2 = 0.4,\ y_2 = 0.04, \qquad m_2 = x_2 + y_2 = 0.44,$$

and so on.

If you have to do more than a very few steps of such a calculation by hand, it is a good idea to lay it out as a table, for example:

| $r$ | $x_r$ | $y_r$ | $m_r = x_r + y_r$ | $hm_r$ | $y_{r+1} = y_r + hm_r$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.2 | 0 | 0.2 | 0.04 | 0.04 |
| 2 | 0.4 | 0.04 | 0.44 | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | | | | |

After each value of $y_{r+1}$ has been calculated from the recurrence relation and entered in the last column, it is transferred to the $y_r$ column in the next row.

**Exercise 2**

Complete the above table and so obtain the coordinates of $P_5$. Check your table by comparing it with your graph from Exercise 1.

[*Solution on p. 44*]

**Exercise 3**

What changes would be necessary in the column headings of the above table if the equation $\frac{dy}{dx} = x + y$ were replaced by

$$\frac{dy}{dx} = y?$$

For this new equation, use the same step length, $h = 0.2$, to calculate an approximation to $y(1)$ given that $y(0) = 1$ (the notation $y(1)$ means 'the value of $y$ when $x = 1$' and $y(0)$ means 'the value of $y$ when $x = 0$').

[*Solution on p. 44*]

## 2.3 Choosing the step length: accuracy versus cost

Whenever we use Euler's method we have to choose a value for the step length $h$; in the last subsection this choice was made so as to give a convenient illustration of the working of the method. When a method such as Euler's is used for the purpose of solving some particular differential equation rather than for illustration, however, we have to consider the choice of step length more carefully. Two opposing requirements, accuracy and cost, influence this choice, and the decision must be a compromise between them.

The accuracy of Euler's method usually improves when we take a smaller step, size. One way of seeing this is to consider the graphical description of Euler's method used in Subsection 2.1. The smaller the step size, the more often we adjust the slope of our polygonal approximation to the trajectory, and so the less the approximation deviates from the trajectory (see Figure 7).

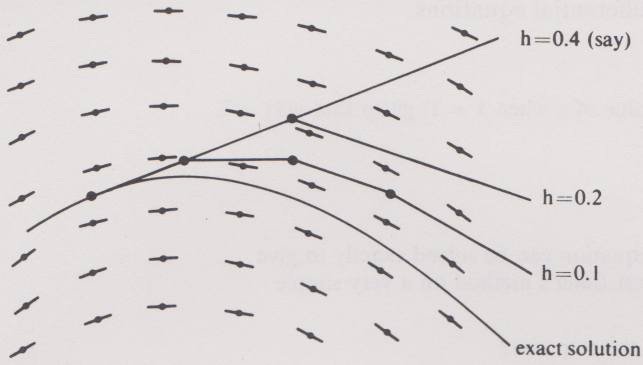



Figure 7

The effect can also be seen by calculation. The next table shows approximate solutions, using different values of $h$, for the problem considered in Exercise 3.

Given that

$$\frac{dy}{dx} = y$$

and $y(0) = 1$, find $y(1)$.

The table also shows the deviations of these approximations from the exact value of $y(1)$, which is $e = 2.718282\ldots$ (you can check that $y = e^x$ is a solution of the differential equation, that it also satisfies the condition $y(0) = 1$, and hence that $y(1) = e^1 = e$). All numbers are given to an accuracy of 6 places of decimals.

| $h$ | approximation to $y(1)$ | deviation | number of steps |
|---|---|---|---|
| 0.1 | 2.593742 | 0.124539 | 10 |
| 0.01 | 2.704814 | 0.013468 | 100 |
| 0.001 | 2.716924 | 0.001358 | 1000 |
| 0.0001 | 2.718146 | 0.000136 | 10000 |

You will see that the difference between the approximation to $y(1)$ and the exact value decreases roughly in proportion to $h$. Indeed, it can be shown that by making $h$ small enough we can make the error as small as we please, in other words, the error approaches the limit zero as $h$ approaches zero. This is true of the Euler approximation for any reasonable differential equation, not just the one considered in the example.

A word of caution is necessary here. These remarks about the error are only valid if we do the arithmetic to enough decimal places; if we are working with a calculator or a computer then the number of decimal places we can use is restricted and rounding errors will be introduced. After a certain point any increase in accuracy brought about by reducing the size of $h$ will be swamped by these rounding errors.

Rounding errors are not the only problem. Before concluding that $h$ should always be taken very small we must also consider the cost of this additional accuracy. The last column of the table illustrates how the number of steps goes up in inverse proportion to the step size (the general formula for number of steps is $(b - a)/h$ where $a$ and $b$ are the initial and final values of $x$). Since the error in Euler's method is approximately proportional to step size it follows that for this method a 10-fold improvement in accuracy is paid for by a 10-fold increase in the number of steps required. For the example above, an accuracy of one part in a million could require something like a million steps (and each step would have to be calculated to a very high accuracy—perhaps one part in $10^{12}$). Even with a high speed computer this would not be a sensible way of going about things; instead one would use one of the more efficient methods described in the unit on numerical solutions of differential equations. Thus Euler's method is not suitable for high accuracy work. Its virtue is rather in its simplicity and its clear illustration of the basic principles of the numerical solution of differential equations.

**Exercise 4**

Consider the problem of calculating $y(2)$ (i.e. the value of $y$ when $x = 2$) given that $y(1) = 2$, for the differential equation

$$\frac{dy}{dx} = x,$$

using Euler's method, for various step sizes. (This equation can be solved exactly to give $y = \frac{1}{2}x^2 + \frac{3}{2}$, but the purpose of the exercise is to test Euler's method on a very simple example.)

(i) If the calculation is to contain $n$ steps, what is the step size?

(ii) Calculate the approximation to $y(2)$ for the step sizes that give (a) 1 step, (b) 2 steps, (c) 4 steps.

(iii) Calculate $y(2)$ exactly and hence find the errors in cases (a), (b), (c) above.

(iv) What do the results of (iii) suggest about the dependence of error on $h$ for this problem?

(v) Assuming your hypothesis in (iv) is correct, how many steps would be required to obtain an accuracy of $10^{-4}$ using Euler's method?

[*Solution on p. 44*]

## Summary of Section 2

Given a differential equation

$$\frac{dy}{dx} = m(x, y)$$

and the condition that $y(x_0) = y_0$, where $x_0$ and $y_0$ are specified, then Euler's method can be used to find $y(b)$, where $b$ is a given value of $x$, as follows.

1. Choose $h = (b - x_0)/n$ where $n$ is the number of steps to be used.
2. For $r = 0, 1, \ldots, n - 1$ use the recurrence relation

   $$y_{r+1} = y_r + hm(x_r, y_r)$$

   where $x_r = x_0 + rh$. Then $x_n = b$ and $y_n \simeq y(b)$.

The error in the approximation to $y(b)$ is *roughly* proportional to $h$, and therefore inversely proportional to the cost (i.e. number of steps) of the calculation.

# 3 Direct integration and separation of variables

## 3.0 Introduction

This section and the one that follows can be studied either before or after Sections 1 and 2. They deal with methods of finding exact solutions, that is solutions expressible as a formula, for certain types of differential equation.

A feature common to all these exact methods is that at some stage they all involve carrying out an integration. We shall see that some types of equation have to be rearranged before this integration can be carried out but we begin by considering some equations which can be solved by carrying out the integration directly.

## 3.1 Direct integration I: general and particular solutions

Here is one example of a differential equation which can be solved exactly:

$$\frac{dy}{dx} = 2x.$$

Before proceeding, see if you can find a solution. Write it in the box below.

$y =$ ☐

As we have already noted in the introduction to this unit, a solution of a differential equation is a function; so we want the box to contain a function of $x$, that is, an expression depending only on $x$. The condition this expression must satisfy is given by the differential equation: it is that differentiation of the expression in the box (with respect to $x$) must give the expression $2x$. In other words, the box must contain a primitive of the function $2x$ (a **primitive** of a function $f$ is any function $F$ whose derivative is $f$). As you probably know one primitive of the function $2x$ is the function $x^2$, and so you could have written $x^2$ in the box, giving $y = x^2$ as a solution of the differential equation.

This is not the only solution, however; you may recall from an earlier course that adding any constant to a primitive gives another primitive. So, for example, the function defined by

$$y = x^2 + 5$$

is another solution of the differential equation. (You should check this last statement.) In fact, any function defined by

$$y = x^2 + C,$$

where $C$ is a constant, is a solution of the differential equation. The two solutions just considered are the cases $C = 0$ and $C = 5$ respectively, but any other numerical value for $C$ also gives a solution.

The method of solution we are using here can be applied to any differential equation of the form

$$\frac{dy}{dx} = f(x)$$

where $f$ is some given function (for example the function $2x$ in the above example). The important thing is that the right-hand side must not involve $y$. All you have to do in such cases is to integrate the function $f$. The solutions of the differential equation are then given by

$$y = \int f(x)\,dx + C$$

where $\int f(x)\,dx$ is one of the primitives of the function $f$, and $C$ is an arbitrary constant.

The appearance of the constant in this solution is a feature common to all differential equations; it expresses the fact that there is an infinite collection of functions which satisfy the differential equation, one for each value of the constant $C$. The collection of all the solutions of a differential equation is called the **family of solutions**.

Just as in the case of recurrence relations, we distinguish between particular and general solutions of a differential equation. For the differential equation $dy/dx = 2x$ studied above, the function defined by $y = x^2$ is a **particular solution**, and the function defined by $y = x^2 + 5$ is another. But a formula such as

$y = x^2 + C$, containing an arbitrary constant, is called the **general solution**. The general solution describes not just one solution but the whole family, from which we can pick out any particular solution by giving the arbitrary constant a particular value.

Depending on the problem giving rise to the differential equation, we may require either the general or a particular solution. If a particular solution is required, then the problem will contain some additional piece of information by means of which this solution can be picked out from the family of all solutions. For example, suppose our problem is:

> Find a solution of the equation
> $$\frac{dy}{dx} = 2x$$
> such that $y(3) = 4$.

The condition $y(3) = 4$, which means 'when $x = 3$, $y = 4$', is the additional piece of information we need in order to pick out the desired particular solution. The way to use this information is first to obtain the general solution of the differential equation, which we already know to be

$$y = x^2 + C,$$

and then use the extra condition $y(3) = 4$ to give an algebraic equation for $C$. Putting $x = 3$ in the general solution we obtain

$$y = 9 + C$$

and so the condition $y(3) = 4$ gives

$$4 = 9 + C.$$

Solving for $C$ gives $C = -5$ and so the required particular solution is

$$y = x^2 - 5.$$

**Exercise 1**

(i) Find the general solution of the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the particular solution for which $y(1) = 5$.

(ii) Find the general solution of the differential equation

$$\frac{dy}{dx} = e^{-2x}$$

and the particular solution for which $y(0) = 2$.

(iii) Find the general solution of the differential equation

$$\frac{dy}{dx} = a \sin bx$$

(where $a$ and $b$ are given constants, with $b \neq 0$) and the particular solution for which $y(0) = 0$.

[*Solutions on p. 45*]

## 3.2 Direct integration II

All the differential equations we have considered so far in this section have the special form

$$\frac{dy}{dx} = f(x)$$

where $f$ is some given function. Although the derivative of $y$ appears in the equation (that is why it is called a *differential* equation) $y$ itself does not appear. In consequence we were able to solve the equation by integrating the function $f$. We can think of this in a slightly different way: in effect, we integrated *both* sides of the equation directly with respect to the same variable $x$, obtaining

$$y = \int f(x)\,dx + C$$

N.B. Always integrate both sides with respect to the same variable.

where $y$ is a primitive of $\dfrac{dy}{dx}$ just as $\int f(x)\,dx$ is a primitive of $f$. (You might think that when integrating both sides of an equation we should also add a constant to the left-hand side. However, adding two constants, one to each side of an equation, is equivalent to adding their difference, $C$ say, to the right-hand side.)

Let us now see whether any more general differential equations, containing $y$ as well as its derivative, can be solved by integrating both sides of the equation. Consider for example, the equation

$$2y\frac{dy}{dx} = -2x.$$

This equation is equivalent to the equation

$$\frac{dy}{dx} = -\frac{x}{y}$$

discussed in the television programme.

We know that the right-hand side can be expressed as the derivative of $-x^2$ with respect to $x$. The remarkable thing about this equation is that the left-hand side, even though it contains $y$ which is related to $x$ in an unknown way, can be expressed as the derivative (with respect to $x$) of a simple expression. Before you read on, try to find a simple expression whose derivative (with respect to $x$) is equal to the left-hand side. (Hint: use the chain rule.)

The required expression is $y^2$; for if we differentiate $y^2$, using the chain rule, we get

$$\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 2y\frac{dy}{dx}.$$

This formula is true whatever functional relation connects $y$ and $x$, so it does not matter that we do not know what this functional relation is.

The above remarks show that the differential equation can be written in the form

$$\frac{d(y^2)}{dx} = \frac{d(-x^2)}{dx}.$$

It follows that $y^2$ and $-x^2$ are both primitives of the same function and must therefore be equal or differ by a constant. We can therefore integrate both sides of the equation to obtain

$$y^2 = -x^2 + C$$

where $C$ is an arbitrary constant. This last formula contains the general solution of our differential equation.

In practice, it is helpful to lay out calculations of this kind as follows. Starting from the original differential equation

$$2y\frac{dy}{dx} = -2x$$

we integrate both sides with respect to $x$, obtaining two primitives which must either be equal or differ by a constant. Using indefinite integral notation for these primitives, the formula expressing this fact is

$$\int \left(2y\frac{dy}{dx}\right) dx = \int (-2x)\,dx + C.$$

Remember: there is no need to include a constant on the left-hand side, one constant is enough.

Even though we do not know yet how $x$ and $y$ are related, and therefore do not know what expressions in $x$ the symbols $y$ and $dy/dx$ stand for, we can use the formula for integration by substitution to simplify the left-hand side. In this way the equation becomes

$$\int 2y\,dy = \int -2x\,dx + C.$$

Now straightforward integration gives

$$y^2 = -x^2 + C.$$

Writing this formula in the form $y^2 + x^2 = C$ shows that the family of solutions can be visualized as concentric circles. This observation is explored further in the television programme.

(There is no need to include arbitrary constants when integrating $2y$ and $-2x$ since they have already been taken care of by the constant $C$.) Since this last

equation shows how the variables $x$ and $y$ are related for each value of $C$, it implicitly describes the family of solutions of our differential equation. However, solutions of differential equations are *functions* and so we must find a formula expressing $y$ *explicitly* as a function of $x$. This can be done by treating the above implicit solution as an algebraic equation and solving it for $y$ to obtain

$$y = \pm\sqrt{C - x^2}.$$

(The $\pm$ sign means that for each value of the constant $C$ there are two distinct functions satisfying the differential equation: $y = \sqrt{C - x^2}$ and $y = -\sqrt{C - x^2}$.) Notice that in this solution the constant of integration $C$ is no longer added to the right-hand side of the equation but has been absorbed under the square root sign. Nevertheless, by giving $C$ its various values we still generate the whole family of solutions. The above formula is therefore the general solution of the differential equation.

Since the road by which we arrive at such solutions is a fairly long one, the results should always be checked by substituting in the original differential equation. In the present case, differentiation gives

$$\frac{dy}{dx} = \mp\frac{x}{\sqrt{C - x^2}}$$

Wherever the symbols $\pm$ or $\mp$ appear, the upper sign refers to the solution $y = \sqrt{C - x^2}$ and the lower sign to the solution $y = -\sqrt{C - x^2}$

so that

$$2y\frac{dy}{dx} = (\pm 2\sqrt{C - x^2}) \times \left(\mp\frac{x}{\sqrt{C - x^2}}\right)$$

$$= -2x,$$

which agrees with our original differential equation.

The procedure described above can be summarized as follows.

**Procedure 3.2**

1. This procedure applies to differential equations which can be written in the form

$$g(y)\frac{dy}{dx} = f(x)$$

where $f$ and $g$ are given functions.

2. Integrate both sides with respect to the same variable $x$ obtaining

$$\int g(y)\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$$

Always be sure to include the arbitrary constant $C$.

3. The rule for integration by substitution justifies writing this as

$$\int g(y)\,dy = \int f(x)\,dx + C.$$

4. Carry out the two integrations, to obtain a formula relating $x$ and $y$. (This assumes, of course, that the integrations can be done.)

5. Rearrange the formula in Step 4 to give $y$ in terms of $x$.

6. Check that the resulting formula for $y$ satisfies the differential equation. If it does then you have found the general solution of your differential equation.

**Exercise 2**

In each of the following, 'solve' means 'find the general solution of'. In part (iv) the condition ($y > 0$) ensures that $\frac{1}{y}$ makes sense; it means that you are only required to find solutions for which $y$ is positive for all values of $x$.

(i) Solve

$$y^2\frac{dy}{dx} = x^4$$

and check your solution.

(ii) Solve

$$e^y\frac{dy}{dx} = \frac{1}{1+x^2}$$

and check your solution.

(iii) Solve

$$\frac{1}{1+y^2}\frac{dy}{dx} = x$$

and check your solution.

(iv) Solve

$$\frac{1}{y}\frac{dy}{dx} = 1 \qquad (y > 0)$$

and check your solution.

[*Solution on p. 45*]

## 3.3 Separation of variables

This subsection is about a type of differential equation which arises very frequently and which can be solved by bringing it to the form considered in the previous subsection. One equation of this type is

$$\frac{dy}{dx} = x(1+y^2). \tag{1}$$

This cannot be integrated directly because the variables $x$ and $y$ appear together on the right-hand side, however, we can bring it to the form

$$g(y)\frac{dy}{dx} = f(x),$$

which can be integrated using Procedure 3.2, by dividing both sides of Equation (1) by $1 + y^2$. This gives

$$\frac{1}{1+y^2}\frac{dy}{dx} = x$$

which is one of the equations you solved in Exercise 2, obtaining the general solution

$$y = \tan(\tfrac{1}{2}x^2 + C).$$

This must also be the general solution of Equation (1) for we do not alter the information contained in Equation (1) by dividing it on both sides by $1 + y^2$.

The procedure we have used here is known as **separation of variables**, because by dividing by $1 + y^2$ we brought the equation to a form where the left-hand side depended only on the variable $y$ and its derivative $dy/dx$, while the right-hand side depended only on the variable $x$: the two variables $x$ and $y$ were *separated* to the two sides of the equation. The procedure can be summarized as follows.

**Procedure 3.3: Separation of variables**

1. This procedure applies to differential equations which can be brought to the form

$$\frac{dy}{dx} = f(x)h(y)$$

where $f$ and $h$ are known functions.

2. Divide both sides by $h(y)$, obtaining

$$\frac{1}{h(y)}\frac{dy}{dx} = f(x).$$

3. Apply Procedure 3.2 to this last equation.

A complication which sometimes arises is that $h(y)$ may take the value zero for some 'critical' value or values of $y$. Since division by zero is meaningless, a solution obtained by the separation of variables method can only be relied on to the extent that it avoids such critical values of $y$.

If there are critical values of $y$ (that is, values of $y$ for which $h(y) = 0$) then there will be solutions of the differential equation which *cannot* be found using the separation of variables method. For example, any constant function, with constant equal to a critical value of $y$, will satisfy the equation by making both $\frac{dy}{dx}$ and $h(y)$ zero, but to find such a solution using separation of variables would involve dividing by $h(y) = 0$. However, there will also be solutions which avoid the critical values of $y$ and they *can* be found using the separation of variables method, as the following example illustrates.

A constant function is one of the form $y = C$ where $C$ is a number; that is, $y$ takes a constant value $C$ for all values of $x$.

**Example 1**

Consider the equation

$$\frac{dy}{dx} = y \tag{2}$$

This has the standard form of Procedure 3.3, with $f(x) = 1$ and $h(y) = y$. In this case the only 'critical value' of $y$ is zero. One solution of the differential equation, not given by the separation of variables method, is therefore the constant function $y = 0$ (that is, the function which takes the value 0 for all values of $x$). All the other solutions can be found using the separation of variables method, but to avoid the critical value 0 we treat solutions for which $y > 0$ and $y < 0$ separately.

Dividing both sides of the equation by $y$ (valid for all non-zero values of $y$) gives

$$\frac{1}{y}\frac{dy}{dx} = 1$$

so that

$$\int \frac{1}{y}\,dy = x + C \tag{3}$$

where $C$ is an arbitrary constant. Now we want a primitive of $1/y$. If $y > 0$ we can use the obvious one, which is $\log_e y$, obtaining

$$\log_e y = x + C \qquad (y > 0).$$

Solving for $y$ then gives

$$y = e^{x+C}.$$

But if $y < 0$ the primitive $\log_e y$ will not do since negative numbers do not have (real) logarithms. The appropriate primitive of $1/y$ for negative $y$ is $\log_e(-y)$. (If you have not come across this before you should check that it does have $1/y$ as its derivative, and also note that $-y$ is positive so that it does have a logarithm.) Using this primitive in Equation (3) gives

$$\log_e(-y) = x + C \qquad (y < 0),$$

and solving for $y$ then gives

$$y = -e^{x+C}.$$

So it seems that the family of solutions of the differential equation requires the three formulae

$$y = e^{x+C}, \qquad y = -e^{x+C}, \qquad y = 0 \tag{4}$$

for its complete description. However, in Section 4 we shall solve the same differential equation by a different method and obtain the general solution as a single formula:

$$y = Ce^x \tag{5}$$

where $C$ is an arbitrary constant. On the face of it this formula looks quite different from Formulae (4) but, as we shall see in Section 4, Formulae (4) and Formula (5) describe the same family of solutions. We shall return to this in the tape commentary for Section 4 but in the meantime you may like to think about it yourself.

**Exercise 3**

Obtain and check the general solutions of the following differential equations. (In some cases the allowed values of $x$ and $y$ have been restricted to ensure that you will not be in danger of dividing by zero.)

(i) $\dfrac{dy}{dx} = e^{x+y}$ Hint: $e^{x+y} = e^x e^y$

(ii) $x\dfrac{dy}{dx} = y \qquad (x > 0, y > 0)$

(iii) $\dfrac{dy}{dx} = \dfrac{x}{y} \qquad (y > 0)$

[*Solution on p. 45*]

**Exercise 4**

Obtain and check the general solution of the following differential equations, which are used in a later unit. In each case $A$ and $B$ are positive constants.

(i) $\dfrac{dy}{dx} = -Ay \qquad (y > 0)$

(ii) $\dfrac{dy}{dx} = -Ay^2 \qquad (y > 0)$

(iii) $A\dfrac{dy}{dx} = B - y \qquad (B - y > 0)$

(iv) $A\dfrac{dy}{dx} = B - y \qquad (B - y < 0)$

(v) $A\dfrac{dy}{dx} = B^2 + y^2$

[*Solution on p. 46*]

## 3.4 Partial fractions

In *Unit 3* we shall need the solution of the following differential equation

$$\frac{dy}{dx} = ay - by^2$$

where $a$ and $b$ are constant. Provided that $ay - by^2 \neq 0$ (that is, $y \neq 0$, $y \neq a/b$) we can use the separation of variables procedure to bring the equation to the form

$$\frac{1}{ay - by^2}\frac{dy}{dx} = 1$$

and integration on both sides then gives

$$\int \frac{dy}{ay - by^2} = x + C$$

where $C$ is an arbitrary constant. But to complete the solution we need to evaluate the integral on the left-hand side, which is of a type you may not have seen before.

As always with integrals, we try to transform the integral to one of the types we know how to deal with. In this case neither integration by parts nor substitution (unless you are extremely clever) is any help; but the following procedure known as a **partial fraction expansion**, solves the problem.
We notice that the integrand has a denominator which can be factorized:

$$\frac{1}{ay - by^2} = \frac{1}{(a - by)y}.$$

The trick is to look for two numbers $N_1$ and $N_2$ with the property that

$$\frac{1}{(a - by)y} = \frac{N_1}{a - by} + \frac{N_2}{y} \qquad (y \neq 0 \text{ and } y \neq a/b). \tag{1}$$

Of course it is not obvious *a priori* that such numbers exist; but if they do then we can multiply both sides of the last equation by $(a - by)y$ to obtain the equivalent condition

$$1 = N_1 y + N_2(a - by) \qquad (y \neq 0 \text{ and } y \neq a/b)$$

that is

$$1 = N_2 a + (N_1 - N_2 b)y \qquad (y \neq 0 \text{ and } y \neq a/b).$$

This condition is satisfied for all values of $y$ (except 0 and $a/b$) if and only if

$$1 = N_2 a \quad \text{and} \quad 0 = N_1 - N_2 b.$$

This last condition can be solved for $N_1$ and $N_2$ to give

$$N_2 = 1/a \quad \text{and} \quad N_1 = N_2 b = b/a.$$

So provided that $a$ is different from zero our supposition that the numbers $N_1$ and $N_2$ exist is justified and we have, from Equation (1)

$$\begin{aligned}\frac{1}{(a-by)y} &= \frac{b/a}{a-by} + \frac{1/a}{y} \\ &= \frac{b}{a(a-by)} + \frac{1}{ay} \qquad (y \neq 0 \text{ and } y \neq a/b).\end{aligned}$$

It is a good idea to check such equations by trying a particular value of $y$; for example, substituting $y = 1$ into the right-hand side gives

$$\begin{aligned}\frac{b}{a(a-b)} + \frac{1}{a} &= \frac{b + (a-b)}{a(a-b)} \\ &= \frac{1}{a-b}\end{aligned}$$

which is the same as the expression obtained by substituting $y = 1$ into the left-hand side.

Now we can do the integration. To avoid the critical values 0 and $a/b$ it is necessary to consider various ranges of $y$ separately. We start by considering the case $y > 0$, $a - by > 0$:

$$\begin{aligned}\int \frac{dy}{ay - by^2} &= \int \left( \frac{b}{a(a-by)} + \frac{1}{ay} \right) dy \\ &= \frac{b}{a} \int \frac{dy}{a-by} + \frac{1}{a} \int \frac{dy}{y} \\ &= -\frac{1}{a} \log_e (a - by) + \frac{1}{a} \log_e y \qquad (\text{since } y > 0,\ a - by > 0)\end{aligned}$$

and so we can finish solving our differential equation. We had already got as far as

$$\int \frac{dy}{ay - by^2} = x + C;$$

we can now write

$$-\frac{1}{a} \log_e (a - by) + \frac{1}{a} \log_e y = x + C.$$

To complete the solution we want to solve for $y$. Multiplying both sides by $a$ and using the properties of logarithms gives

$$\log_e \frac{y}{a - by} = a(x + C)$$

taking the exponential of both sides and multiplying by $a - by$ gives

$$y = (a - by) \exp [a(x + C)]$$

hence

$$y(1 + b \exp [a(x + C)]) = a \exp [a(x + C)].$$

Finally we have

$$y = \frac{a \exp [a(x + C)]}{1 + b \exp [a(x + C)]},$$

a solution which is valid provided $y > 0$ and $a - by > 0$, that is, provided $0 < y < a/b$. Other ranges of $y$ can be treated by the same method.

A similar method can be used for any integral where the integrand is the ratio of two polynomials. The procedure described below covers only the simplest cases to which the method applies. One example of a more complicated case (with a cubic in the denominator) is given in Exercise 5(iv).

**Procedure 3.4: Partial fractions**

To express in partial fraction form the function

$$\frac{\alpha x + \beta}{ax^2 + bx + c}$$

where $a$, $b$, $c$, $\alpha$, $\beta$ are numbers (with $a \neq 0$) such that the denominator can be factorized into *distinct* factors. This condition will be satisfied if $b^2 - 4ac > 0$; for then the quadratic equation $ax^2 + bx + c = 0$ will have two distinct real roots.

1. Let $x_1$, $x_2$ be the solutions of $ax^2 + bx + c = 0$. Then the function becomes

$$\frac{\alpha x + \beta}{a(x - x_1)(x - x_2)}.$$

2. Suppose that numbers $N_1$, $N_2$ exist such that

$$\frac{\alpha x + \beta}{a(x - x_1)(x - x_2)} = \frac{N_1}{x - x_1} + \frac{N_2}{x - x_2} \quad (x \neq x_1 \text{ and } x \neq x_2).$$

3. Multiply both sides by $a(x - x_1)(x - x_2)$ and collect terms:

$$\begin{aligned}\alpha x + \beta &= aN_1(x - x_2) + aN_2(x - x_1) \\ &= a(N_1 + N_2)x - a(N_1x_2 + N_2x_1)\end{aligned}$$

4. Since this last equation holds for all $x$ (except $x_1$ and $x_2$) we must have

$$\alpha = aN_1 + aN_2$$
$$\beta = -ax_2N_1 - ax_1N_2.$$

5. Solve the above pair of simultaneous equations for $N_1$ and $N_2$.

6. Check that your result

$$\frac{\alpha x + \beta}{ax^2 + bx + c} = \frac{N_1}{x - x_1} + \frac{N_2}{x - x_2}$$

is correct by substituting one or two particular values of $x$ (not $x_1$ or $x_2$).

This procedure does not cover the cases where the quadratic has one or no real roots. If it has one real root, say $x_1$, then the substitution $x - x_1 = w$ will simplify the integral. If it has no real roots then a substitution involving arc tan will simplify the integral (cf. solution to Exercise 4(v)). But such integrals are beyond the scope of this section.

**Exercise 5**

Find partial fraction expansions for

(i) $\dfrac{1}{x^2 + x}$

(ii) $\dfrac{x + 1}{x^2 + x - 2}$

(iii) $\dfrac{1}{B^2 - y^2}$ (where $B$ is a positive constant)

(iv) $\dfrac{1}{x(x^2 - 1)}$

[*Solution on p. 47*]

**Exercise 6**

Using the partial fraction expansions you found in Exercise 5, solve the differential equations

(i) $\dfrac{dy}{dx} = y^2 + y \qquad (y > 0)$

(ii) $\dfrac{dy}{dx} = \dfrac{x+1}{x^2+x-2} \qquad (-2 < x < 1)$

(iii) $A\dfrac{dy}{dx} = B^2 - y^2 \qquad (-B < y < B)$

You will meet this equation again in *Unit 4*

[*Solution on p. 47*]

## Summary of Section 3

**Differential equations**

The solution of the equation

$$\frac{dy}{dx} = f(x)$$

with $f$ a given function, is

$$y = \int f(x)\,dx + C$$

where $C$ is an arbitrary constant.

To solve the equation Procedure 3.2.

$$g(y)\frac{dy}{dx} = f(x)$$

make $y$ the subject of the formula

$$\int g(y)\,dy = \int f(x)\,dx + C.$$

To solve the equation Procedure 3.3.

$$\frac{dy}{dx} = f(x)h(y) \qquad (h(y) \neq 0)$$

make $y$ the subject of the formula

$$\int \frac{1}{h(y)}\,dy = \int f(x)\,dx + C.$$

The above are general solutions involving a constant $C$; to obtain the particular solution satisfying a given further condition, use the further condition to evaluate $C$.

**Integrals**

$$\int \frac{dx}{ax+b} = \begin{cases} \dfrac{1}{a}\log_e(ax+b) & \text{if } ax+b > 0 \\ \dfrac{1}{a}\log_e(-ax-b) & \text{if } ax+b < 0 \end{cases}$$

To integrate the function

$$\frac{\alpha x + \beta}{ax^2 + bx + c}$$

where $b^2 > 4ac$, first express it in partial fractions: Procedure 3.4.

$$\frac{N_1}{x - x_1} + \frac{N_2}{x - x_2}$$

where $N_1$, $N_2$ are constants and $x_1$, $x_2$ are the roots of the quadratic $ax^2 + bx + c$.

# 4 Linear equations

## 4.0 Introduction

There is no universal method for obtaining the general solution of a differential equation, but the method to be described in this section, the *integrating factor method*, does solve an important class of equations, called *linear equations*, for which the separation of variables method may not work. Like the separation of variables method, the integrating factor method depends on rearranging the differential equation to a form that can be integrated directly. We begin by discussing the type of direct integration appropriate for this.

## 4.1 Direct integration using the product rule

The form of direct integration considered here is based on the formula for differentiating a product. The product we are interested in is $p(x)y$, where $p$ is some function to be chosen later. The formula for differentiating $p(x)y$ is

$$\frac{d}{dx}[p(x)y] = p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y.$$

Consequently, if the differential equation we are trying to solve can be written in the form

$$p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y = q(x), \tag{1}$$

for some functions $p$ and $q$, then the differential equation is equivalent to

$$\frac{d}{dx}[p(x)y] = q(x)$$

and can therefore be integrated directly, to give

$$p(x)y = \int q(x)\,dx + C.$$

Dividing through by $p(x)$ then gives $y$ as a function of $x$.

So any differential equation that can be brought to the form of Equation (1) can be integrated directly. The next subsection describes how this can be done for certain equations.

**Exercise 1**

Check that the following differential equations have the form of Equation (1). Determine $p$ and $q$ in each case, and hence integrate the equations.

(i) $x\dfrac{dy}{dx} + y = 1$

(ii) $\sin x\dfrac{dy}{dx} + (\cos x)y = \sin x$

(iii) $e^{3x}\dfrac{dy}{dx} + 3e^{3x}y = e^{2x}$

[*Solution on p. 48*]

## 4.2 The integrating factor method

The method we are going to study here applies to any differential equation of the form

$$\frac{dy}{dx} = k(x) + l(x)y \tag{2}$$

where $k$ and $l$ are given functions. An equation having this form is said to be **linear**.

There is an analogy here with linear first order recurrence relations, that is, recurrence relations of the form

$$u_{r+1} = k_r + l_r u_r.$$

To solve Equation (2) we want to find a way of rearranging it to the form

$$p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y = q(x) \tag{1}$$

which we saw how to integrate in the previous subsection. (We do not yet know what the functions $p$ and $q$ are, but we shall find out during the course of the calculation.) As a first step towards bringing Equation (2) to the form of Equation (1), we subtract $l(x)y$ from both sides of Equation (2), obtaining

$$\frac{dy}{dx} - l(x)y = k(x). \tag{3}$$

To make the first term of this equation match the first term of Equation (1) we multiply both sides by $p(x)$, to get

$$p(x)\frac{dy}{dx} - p(x)l(x)y = p(x)k(x). \tag{4}$$

The remaining terms of this equation can be made to match Equation (1) by choosing the functions $p$ and $q$ so that

$$\frac{dp(x)}{dx} = -p(x)l(x) \tag{5}$$

and

$$q(x) = p(x)k(x).$$

In fact these two conditions are sufficient to determine the two functions $p$ and $q$. We are particularly interested in the function $p$, because multiplication by $p(x)$ takes us from Equation (3), which is not of the directly integrable form, to Equation (1), which is. For this reason $p(x)$ is called an **integrating factor**.

The condition determining $p(x)$ is Equation (5), a differential equation which can be solved by the method of separation of variables. Dividing both sides by $p(x)$ and then integrating both sides gives

$$\int \frac{1}{p(x)}\frac{dp(x)}{dx}dx = \int(-l(x))\,dx$$

that is

$$\int \frac{dp(x)}{p(x)} = \int(-l(x))\,dx$$

and so one solution is given by

$$\log_e p(x) = \int(-l(x))\,dx$$

hence

$$p(x) = \exp\left[\int(-l(x))\,dx\right]. \tag{6}$$

No arbitrary constant is necessary, nor need we look for solutions with $p(x) < 0$, because we only need *one* way of bringing our differential equation to the form of Equation (1), not all the possible ways. Having determined the integrating factor $p(x)$ from Equation (6), we can solve Equation (4) (and hence our original differential equation) by direct integration, after which it is easy to find $y$. The following example should clarify the procedure.

**Example 1**

Let us apply the integrating factor method to the equation

$$\frac{dy}{dx} = x + y. \tag{7}$$

Equation (7) is discussed in the television programme for this unit

This has the linear form of Equation (2), with $k(x) = x$ and $l(x) = 1$. The first step is to subtract the term containing $y$, which in this case is $y$ itself, from both sides, obtaining

$$\frac{dy}{dx} - y = x. \tag{8}$$

Now we want the integrating factor $p(x)$. This is best found from Equation (6):

$$\begin{aligned} p(x) &= \exp\left[\int(-l(x))\,dx\right] \\ &= \exp\left[\int(-1)\,dx\right] \\ &= e^{-x}. \end{aligned}$$

Multiplying both sides of Equation (8) by this integrating factor we obtain

$$e^{-x}\frac{dy}{dx} - e^{-x}y = e^{-x}x. \tag{9}$$

If the integrating factor has been calculated correctly, the left-hand side of Equation (9) should be the derivative of the product $p(x)y$, that is of $e^{-x}y$. Using the product rule to calculate the derivative of $e^{-x}y$ we find that Equation (9) can indeed be written

$$\frac{d}{dx}[e^{-x}y] = e^{-x}x.$$

Direct integration now gives

$$\begin{aligned} e^{-x}y &= \int e^{-x}x\,dx \\ &= -xe^{-x} + \int e^{-x}\,dx \qquad \text{(integrating by parts)} \\ &= -xe^{-x} - e^{-x} + C. \end{aligned}$$

Finally we solve for $y$, to obtain the general solution of our Differential Equation (7):

$$y = Ce^{x} - x - 1.$$

The solution given in the introduction to this unit is a particular case of this, with $C = 1$.

The general procedure for solving a first order differential equation by the integrating factor method is given as Procedure 4.2 on the next page.

**Procedure 4.2: The integrating factor method**

1. This procedure applies to differential equations which can be written in the form

$$\frac{dy}{dx} = k(x) + l(x)y$$

where $k$ and $l$ are given functions; such equations are said to be linear.

2. Subtract $l(x)y$ from both sides, obtaining

$$\frac{dy}{dx} - l(x)y = k(x).$$

3. Define a function $p$ (the integrating factor) by

$$p(x) = \exp\left[\int(-l(x))\,dx\right]$$

(no arbitrary constant is necessary).

4. Multiply both sides of the equation in Step 2 by $p(x)$, obtaining

$$p(x)\frac{dy}{dx} - p(x)l(x)y = p(x)k(x).$$

5. Verify that the left-hand side is the derivative of $p(x)y$ (if not, you have made a mistake in calculating the integrating factor) and integrate both sides, obtaining

$$p(x)y = \int p(x)k(x)\,dx + \text{Constant}.$$

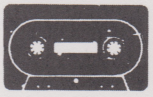
Do *not* forget the arbitrary constant.

6. Carry out the integration of $p(x)k(x)$.

7. Solve algebraically for $y$ (that is, divide both sides by $p(x)$).

8. Check that the expression you have obtained for $y$ satisfies your original differential equation. If it does, you have found the general solution of your differential equation.

## 4.3 Using the integrating factor method (Tape Subsection)



The purpose of this subsection is to give you practice in using the integrating factor method. It consists of three examples accompanied by a tape commentary.

The first example, the equation $\frac{dy}{dx} = y$, has already been solved in Subsection 3.3 using the separation of variables method; but the method used here is better because it covers the cases $y > 0$, $y = 0$ and $y < 0$ all together instead of separately.

While working through the examples you will need to refer to the steps in Procedure 4.2.

*Start the tape when you are ready.*

## 1 A worked example

step 1 → $\frac{dy}{dx} = y$

step 2 → $\frac{dy}{dx} - y = 0$

| $k(x) =$ |
| --- |
| $l(x) =$ |

## 2 The integrating factor

Step 3 → $p(x) = \exp\left[\int(-l(x))\,dx\right]$

$= \exp\left[\int(- \quad)\,dx\right]$

$= \exp\left[\quad\right]$

$= e^{(\quad)}$

$p'(x) =$

differentiate to find $p'(x)$

## 3 Rearranging and integrating

Step 4 → $e^{-x}\frac{dy}{dx} - e^{-x}y = 0$

$\frac{d}{dx}(e^{-x}y) = e^{-x}\frac{dy}{dx} - e^{-x}y$

Step 5 → $e^{-x}y = \int 0\,dx$

$= A$

Step 6

## 4 Solve and check

Step 7 → $y =$

Step 8 → $\frac{dy}{dx} =$ $= y$?

Always check!

## 5 Less help

$$\frac{dy}{dx} + 3y = 2$$

step 1 → $\frac{dy}{dx} =$

$k(x) =$

$l(x) =$

step 2 →

step 3 →

$P(x) = \exp\left[\int -(\quad)\,dx\right]$

$= \exp[\quad]$

$= e^{(\quad)}$

step 4 →

step 5 → $= \int \quad dx$

$=$ ← step 6

$y =$ ← step 7

## 6 Checking

$\frac{dy}{dx} =$

$2 - 3y =$

Equal ?

**7** **On your own**

$$x\frac{dy}{dx} - y = x \qquad x > 0$$

manipulate + identify steps 1, 2 →

$\frac{dy}{dx} =$

$K(x) =$

$l(x) =$

Find p(x) step 3 →

$P(x) = \exp\left[\int -(\quad)\,dx\right]$

$= \exp\left[\quad\right]$

$\exp[-\log_e x] = \exp[\log_e \frac{1}{x}] = \frac{1}{x}$

manipulate steps 4,5 →

Integrate step 6 →

Solution step 7 → $y = x\log_e x + Ax$

Check step 8 →

$\frac{dy}{dx} =$

$x\frac{dy}{dx} - y =$ $= x?$

**8** **Solution? to** $\frac{dy}{dx} = y$

Solution by integrating factor method. → $y = Ae^x$

Solution by separation of variables method →

$y = e^{x+C}$ $(y > 0)$

$y = -e^{x+C}$ $(y < 0)$

$y = 0$

**Solution to $\frac{dy}{dx} = y$**

At the end of the tape commentary we asked you to reconcile the general solution

$$y = Ae^x \tag{10}$$

obtained by the integrating factor method, with the general solution

$$y = e^{x+C}, \qquad y = -e^{x+C}, \qquad y = 0 \tag{11}$$

obtained using separation of variables. To do this observe that positive values of the arbitrary constant $A$ in Solution (10) can equally well be written $e^C$ to give the first of Formulae (11), whereas negative values of $A$ can be written as $-e^C$ to give the second of Formulae (11). Setting $A = 0$ gives the remaining solution $y = 0$.

The above example illustrates the fact that it is often possible to find a more economical way of writing the general solution of a differential equation simply by redefining the arbitrary constant.

### Summary of Section 4

To solve the linear equation

$$\frac{dy}{dx} = k(x) + l(x)y$$

subtract $l(x)y$ from both sides and then multiply both sides by the integrating factor

$$p(x) = \exp\left[\int(-l(x))\,dx\right].$$

The resulting equation has the form

$$p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y = q(x)$$

and integrates directly to give the general solution

$$p(x)y = \int q(x)\,dx + C.$$

# 5 Revision

## 5.1 Choosing the best method

This unit contains a number of different methods for solving differential equations. You will want to know how one decides which to use.

The first step is to bring the equation to the standard form

$$\frac{dy}{dx} = m(x, y).$$

Not every differential equation can be brought to this form, for example the equation may include second or higher derivatives, but such equations are beyond the scope of this unit.

Equations containing second order derivatives are discussed in the next unit on differential equations.

Next look to see if an exact solution is possible by one of the methods described in Sections 3 and 4. In effect, these methods cover two cases only:

(i) equations soluble by the separation of variables method (Procedure 3.3),

$$\frac{dy}{dx} = f(x)h(y),$$

(ii) linear equations using the integrating factor method (Procedure 4.2),

$$\frac{dy}{dx} = k(x) + l(x)y,$$

where $f$ and $h$, or $k$ and $l$, are known functions. The various cases of direct integration can be regarded as special cases of (i) and (ii) (for example, setting $h(y) = 1$ or $l(x) = 0$ gives the direct integration considered at the beginning of Section 3).

If the exact methods fail, either because $m(x, y)$ has the wrong form or because you cannot do the integrations, or even because (as can easily happen) the formula given by the exact solution method is too complicated to be useful, you will have to fall back on an approximate method. This usually means a numerical method and for the time being, therefore, the Euler method. By taking a fairly large step size you can carry out the Euler method using a calculator, but if you want anything better than a qualitative picture of the solution, you will have to use the computer.

A disadvantage of the Euler method (or any other numerical method) is that it only gives particular solutions. If you require information about the general solution for a differential equation you cannot solve exactly, you can obtain it by plotting the direction field and then sketching enough solution curves to give a good picture of their typical behaviour.

**Exercise 1**

For each of the following equations, select the option(s) which give(s) the most appropriate method(s) of solution.

**A** direct integration

**B** separation of variables

**C** the integrating factor method

**D** none of these

(i) $\dfrac{dy}{dx} = x$

(ii) $\dfrac{dy}{dx} = y$

(iii) $\dfrac{dy}{dx} = xy$

(iv) $\dfrac{dy}{dx} = x - y$

(v) $\dfrac{dy}{dx} = y^2$

(vi) $\dfrac{dy}{dx} = x + y^2$

(vii) $\dfrac{dy}{dx} = x^2 + y$

(viii) $\dfrac{dy}{dx} = \sqrt{1 + y}$

(ix) $\dfrac{dy}{dx} = \sqrt{x + y}$

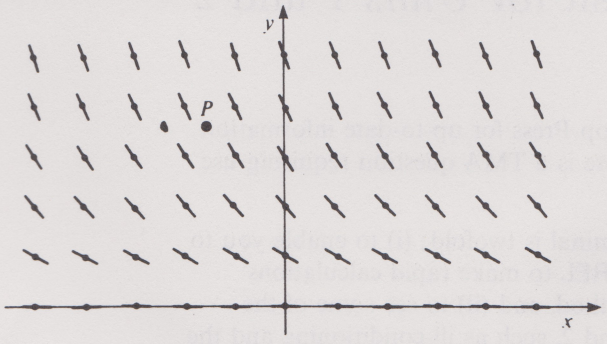
[*Solution on p. 48*]

## 5.2 Further exercises

The exercises in this subsection will enable you to revise and consolidate what you have learned from the rest of the unit.

**Exercise 2**

For the direction field shown, sketch the trajectory through the point $P$.




[*Solution on p. 48*]

**Exercise 3**

For the differential equation

$$\frac{dy}{dx} = 1 - xy$$

with $y(0) = 1$,

(i) write down the recurrence relation for Euler's method with step length 0.1;

(ii) using step length 0.1 calculate the Euler's method approximation to $y(0.3)$.

[*Solution on p. 48*]

**Exercise 4**

If the step size in the above calculation were reduced to 0.05,

(i) would you expect the error in $y(0.3)$ to increase or decrease, and by what factor (roughly)?

(ii) would the labour of calculation increase or decrease and by about what factor?

[*Solution on p. 48*]

**Exercise 5** (*Do not attempt this exercise before studying Section 6.*)

If the step size were further reduced to 0.001, write down the options and responses you would use to carry out the calculation using the computer program RECREL with print-out of the approximations to $y(0.1)$, $y(0.2)$ and $y(0.3)$ only.

[*Solution on p. 48*]

**Exercise 6**

The general solution of a certain differential equation is

$$y = Cx + x^2.$$

Find the particular solution for which $y(4) = 8$.

[*Solution on p. 48*]

**Exercise 7**

Find general solutions of the differential equations

(i) $\dfrac{dy}{dx} + \dfrac{y^2}{x} = 0 \qquad (x > 0, y > 0)$

(ii) $\dfrac{dy}{dx} + \dfrac{2y}{x} = x \qquad (x > 0)$

(iii) $\dfrac{dy}{dx} = y + y^2 \qquad (y > 0)$

(iv) $\dfrac{dy}{dx} = 2y - x$

[*Solution on p. 48*]

**Exercise 8**

Carry out the integration

$$\int \frac{x+4}{x^2+3x+2}\,dx \qquad (x < -2).$$

[*Solution on p. 49*]

# 6 Computer terminal visit for *Units 1* and *2*

## 6.0 Introduction

See the course guide and possibly also the Stop Press for up-to-date information about this visit. Be sure to check whether there is a TMA question requiring use of the terminal.

The purpose of this visit to the computer terminal is twofold; (i) to enable you to learn how to use the computer package RECREL to make rapid calculations involving recurrence relations and Euler's method, and (ii) to see some of the numerical phenomena described in *Units 1* and *2*, such as ill-conditioning and the balance between accuracy and cost, in an actual computation.

Before going to the terminal you should have prepared yourself by studying Section 6 of *Unit 1* and this section. Section 6 of *Unit 1* described in general terms how to work the computer packages, and showed you how to turn algebraic expressions into valid input expressions for the computer. This section contains a description of the package RECREL and some home exercises to help you familiarize yourself with it. After completing the home exercises you will be ready to start planning your work at the terminal based on the computer exercises in Subsection 6.3. It is essential that you prepare these computer exercises carefully at home so that you will not waste time at the terminal.

## 6.1 The computer package RECREL

The computer program RECREL is a package to use when solving first and second order recurrence relations. The package can also be used to implement Euler's method to solve first order differential equations. An example of the use of this package was given on p. 44 of *Unit 1*. You may need to refer to this to help you assimilate the package.

**Command options**

OPTIONS — this tells the program to print a list of available options.

SOLVE — this tells the program to run the problem after all the data has been input. If some data is missing an error message will be printed.

HELP — this tells the program that you are stuck and need advice.

LIST — this tells the program to print the problem and data previously input so that you can check which problem you are solving.

STOP — this is the only way to stop the program.

ROOTS — for constant coefficient second order recurrence relations this option tells the program to compute and print the two roots of the auxiliary equation.

ANSWER — this gives the solution to the computing exercises so that you can check that your answer is correct. To the question

EXERCISE?

you respond with the number of the exercise you are interested in.

**Problem options**

**10 Order of the recurrence relation**

To the question

ORDER = ?

you respond with either 1 or 2 depending on the order of the recurrence relation that you wish to examine. This option is *not* essential for Euler's method.

**11 Input recurrence relation**

To the question

U(R + 1) = ?

you respond by typing in the right hand side as a valid input expression (see Subsection 6.3 of *Unit 1*). For example, if you wish to examine the recurrence relation

$$u_{r+1} = 3u_r - 2u_{r-1} + 3r$$

you would respond by typing

3*U(R) − 2*U(R − 1) + 3*R.

**12 Input $m(x, y)$ for Euler's method**

To the question

Y′ = ?

you respond with a valid input expression for $m(x, y)$. For example if the differential equation were

$$\frac{dy}{dx} = xy + 3$$

your reply to Y′ = ? would be

X*Y + 3.

$y'$ is another way of writing $\frac{dy}{dx}$

**Method options**

**20 Forward recurrence**

This method uses forward recurrence on either a first or second order recurrence relation. The number of terms in the sequence is specified using Option 35.

**21 Backward recurrence**

This method uses backward recurrence on a linear first order recurrence relation using Option 32 to specify the starting point. The program will automatically adapt the recurrence relation specified by Option 11. For example if you want to examine the recurrence relation

$$u_{r+1} = 1 - (r+1)u_r$$

then you input this recurrence relation using Option 11. The package will *automatically* convert this into the form

$$u_r = \frac{1 - u_{r+1}}{r+1}$$

for use with backward recurrence.

**22 Euler's method**

This uses Euler's method to solve the differential equation

$$\frac{dy}{dx} = m(x, y)$$

where $m(x, y)$ is specified using Option 12.

**Parameter options**

**30 Initial condition $u_0$**

This parameter specifies the initial condition $u_0$ for first and second order recurrence relations.

**31 Second initial condition $u_1$**

This parameter specifies the second initial condition for second order recurrence relations.

**32 $u_N$ for backward recurrence**

This is used to input the initial condition for backward recurrence with a linear first order recurrence relation. $N$ is specified by Option 35.

**33 $x_0$ for Euler's method**

This is used to input the initial value of $x$ for Euler's method.

**34 $y_0$ for Euler's method**

This is used to input the initial value of $y$ for Euler's method.

**35 The number of terms $N$ in the sequence**

This is used to input the number of terms in the sequence. This must be specified for both forward and backward recurrence and for Euler's method.

**36 Step size $h$ for Euler's method**

For Euler's method this option specifies the step size.

**Print options**

**40 Outline printout**

If you only want selected output you should use this option to specify $k$ such that only every $k$th term in the sequence is printed.

**41 Full printout (default)**

All terms in the sequence will be printed.

A **default option** is the one the program will use if no other is specified

### 42 Print solution only

Only the last term in the sequence is printed (intermediate terms will not be printed). For backward recurrence the value of $u_0$ will be printed.

## 6.2 Home exercises

The following exercises are to enable you to familiarize yourself with RECREL so that you will not waste time at the terminal.

**Exercise 1**

(i) It is proposed to put the calculation described in Subsection 2.2 on to the computer; that is, to calculate $y(1)$ given that

$$\frac{dy}{dx} = x + y$$

and $y(0) = 0$, with $h = 0.2$. Write in the table the options you would use and the responses to each (where called for) in order to input this problem and print the calculated values of $y(0.2), y(0.4), \ldots, y(1)$.

| Information | Option | Response |
|---|---|---|
| Formula for $m(x, y)$ | | |
| Method | | |
| Intitial value of $y$ | | |
| Initial value of $x$ | | |
| Number of steps | | |
| Step size | | |
| Type of printout | | |

(ii) Suppose you wanted to increase the accuracy by a factor of 100, so that $h$ is reduced to 0.002; but that to save printing time you only wanted to print approximations to $y(0)$, $y(0.1)$, $y(0.2), \ldots, y(1)$. What changes to the answer to (i) would this entail?

[*Solution on p. 50*]

**Exercise 2**

In Sections 3 and 4 of *Unit 1* we considered the recurrence relation

$$u_{r+1} = 1 - (r + 1)u_r$$

with $u_0 = 1 - e^{-1} = 0.63212056\ldots$, which can be used to calculate a sequence of values for the integral

$$u_n = \int_0^1 e^{x-1} x^n \, dx.$$

Make a table, like the one in Exercise 1, showing the information you must give the computer, the options you would use, and the responses to those options, if you wanted to calculate the first 25 terms of the sequence, using

(i) forward recurrence

(ii) backward recurrence, starting at $u_{35}$

[*Solution on p. 50*]

## 6.3 Computer exercises

Now here are the exercises for you to try on the computer. Before setting out, decide on the order in which you will try them; then decide, for each problem, which options you will use and which responses. In this way you will make the best use of your very limited time at the terminal. Solutions to these exercises can be found from the package using the command option ANSWER

*Note*: There may also be a TMA question to be done at the terminal.

*Exercises 3–6 repeat the same numbered exercises from Subsection 6.4 in Unit 1, and are included here to refresh your memory. Exercises 7 and 8 you have not met before.*

**Exercise 3**

Use the package to carry out the problem described in Exercise 2 (use both forward and backward recurrence).

**Exercise 4**

The recurrence relation

$$u_{r+1} = (r+1)u_r - 1$$

with $u_0 = e - 1 = 1.7182818\ldots$ can be used to evaluate a sequence of values for

$$u_n = \int_0^1 x^n e^{1-x}\,dx.$$

Use the package to compute the first 25 terms of this sequence.

**Exercise 5**

Check that your mortgage repayments are correct so that you repay the loan in the specified time period. (Note that your mortgage may include property and/or life insurance which are extras and should be omitted from the calculations.)

If you do not have a mortgage, work out the monthly payment for a £30,000 mortgage borrowed over 25 years if the interest rate is 17%. Use the package to determine the amount outstanding at the end of each year.

**Exercise 6**

Use the package to determine the behaviour of the sequences generated by the following second order recurrence relations.

(i) $u_{r+1} = u_r - u_{r-1}$, $\quad u_0 = 0, u_1 = 1$

(ii) $u_{r+1} = u_r - 2u_{r-1}$, $\quad u_0 = 0, u_1 = 1$

(iii) $u_{r+1} = 0.9u_r - 0.2u_{r-1} + 1500$, $\quad u_0 = 3600, u_1 = 4000$

(iv) $u_{r+1} = 8u_r - 16u_{r-1} + 4$, $\quad u_0 = \frac{4}{9}, u_1 = \frac{4}{9}$.

**Exercise 7**

Use the package to carry out the problems described in Exercise 1 (i) and (ii).

**Exercise 8**

This exercise suggests a possible way of treating problems involving differential equations which you can not solve exactly.

(i) Obtain a numerical solution for the reflector equation

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$$

which satisfies $y(1) = 0$, using a convenient step size, say $h = 0.2$. Take your solution as far as $x = 5$.

The reflector problem was treated in the television programme.

(ii) Plot $y$ against $x$ for $1 \leq x \leq 5$.

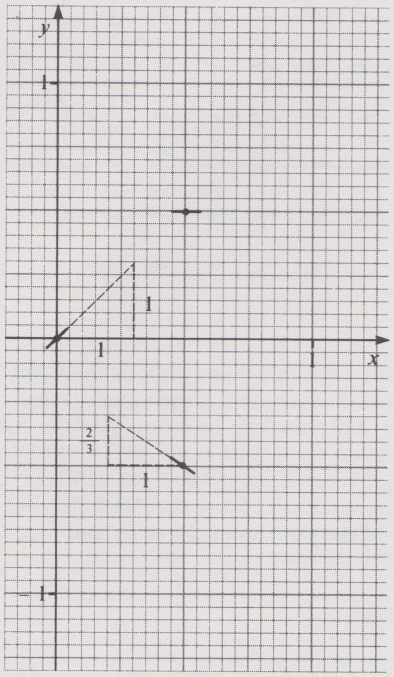
(iii) Try some other step sizes, including $h = 0.1$, $h = 0.05$, $h = 0.02$, and plot a graph to show how the computed value of $y(5)$ varies with $h$. Estimate the exact value of $y(5)$ from this graph.

(iv) (*Optional*) What would happen if you used a negative $h$, with the same initial condition $y(1) = 0$?

# Appendix: Solutions to the exercises

## Solutions to the exercises in Section 1

**1.**




The hypotenuse of the right-angled triangles give the slope of the line segments.

**2.** In each case we start by differentiating both sides of the given formula

(i) $\dfrac{dy}{dx} = \dfrac{d}{dx}(\frac{1}{2}x^2 + 5) = \frac{1}{2}(2x) + 0 = x.$

(ii) $\dfrac{dy}{dx} = \dfrac{d}{dx}(\frac{1}{2}x^2 + C) = \frac{1}{2}(2x) + 0 = x.$

(iii) We have

$$\frac{dy}{dx} = \frac{d}{dx}(Ce^x - x - 1) = Ce^x - 1$$

also $x + y = x + (Ce^x - x - 1) = Ce^x - 1$

so $\dfrac{dy}{dx} = x + y.$

**3.** (i) Using the suggested systematic procedure we have

(a) $\dfrac{dy}{dx} = \dfrac{d}{dx}(R^2 - x^2)^{1/2} = \frac{1}{2}(R^2 - x^2)^{-1/2} \times (-2x)$

$$= \frac{-x}{(R^2 - x^2)^{1/2}}$$

(b) $$-\frac{x}{y} = \frac{-x}{(R^2 - x^2)^{1/2}}$$

(c) so $\dfrac{dy}{dx} = -\dfrac{x}{y}.$

(ii) The procedure used in part (i) gives

(a) $\dfrac{dy}{dx} = \dfrac{x}{C} + 0 = \dfrac{x}{C}$

(b) $$\frac{y + \sqrt{x^2 + y^2}}{x} = \frac{\frac{x^2}{2C} - \frac{C}{2} + \sqrt{x^2 + \left(\frac{x^2}{2C} - \frac{C}{2}\right)^2}}{x}$$

$$= \frac{\frac{x^2}{2C} - \frac{C}{2} + \sqrt{\frac{x^4}{4C^2} + \frac{x^2}{2} + \frac{C^2}{4}}}{x}$$

$$= \frac{\frac{x^2}{2C} - \frac{C}{2} + \sqrt{\left(\frac{x^2}{2C} + \frac{C}{2}\right)^2}}{x}$$

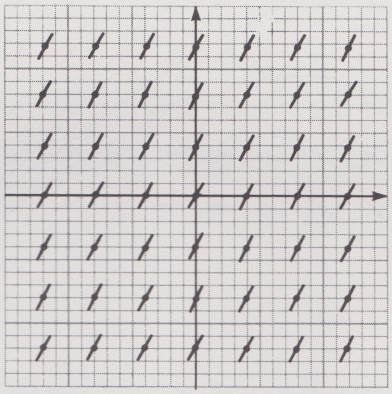
$$= \frac{\frac{x^2}{2C} - \frac{C}{2} + \frac{x^2}{2C} + \frac{C}{2}}{x}$$

$$\left(\text{since } \frac{x^2}{2C} + \frac{C}{2} \geq 0\right)$$

$$= \frac{x}{C}.$$

(c) so $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$

**4.**



From the diagram, the required family of curves appear to be straight lines with slope 2, so the family can be described by the equation

$y = 2x + C.$

Differentiation gives $\dfrac{dy}{dx} = 2 + 0 = 2$, which checks.

**5.** The function giving $y$ on the lower half of the circle is

$y = -\sqrt{R^2 - x^2}.$

Hence the slope of the circle at $(x, y)$ is

$$\frac{dy}{dx} = -\tfrac{1}{2}(R^2 - x^2)^{-1/2} \times (-2x)$$

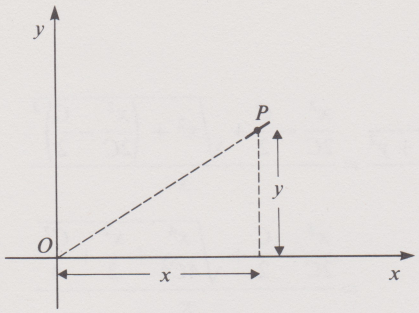
$$= \frac{1}{2y} \times (-2x) = -\frac{x}{y}.$$

**6.** In each case the differential equation is found by equating $\dfrac{dy}{dx}$ with the slope of the direction field at $(x, y)$

(i) $\dfrac{dy}{dx} = -\dfrac{x}{y}$

(ii) $\dfrac{dy}{dx} = 2$

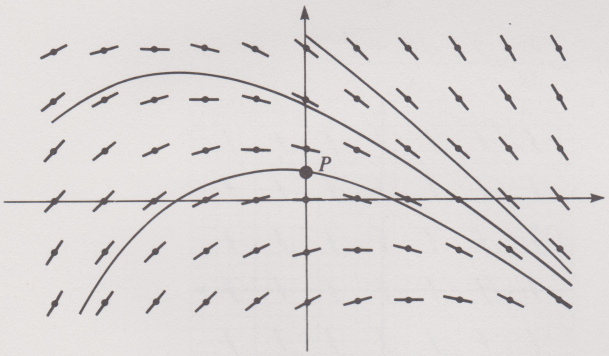
(iii)



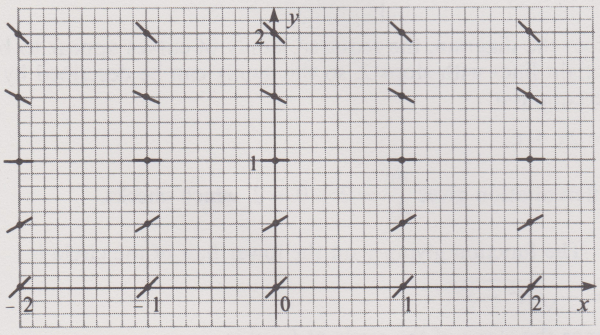

the line $OP$ has slope $\frac{y}{x}$ and so the direction field at $P$ also has slope $\frac{y}{x}$. Hence the differential equation is

$$\frac{dy}{dx} = \frac{y}{x}.$$

7.




8.

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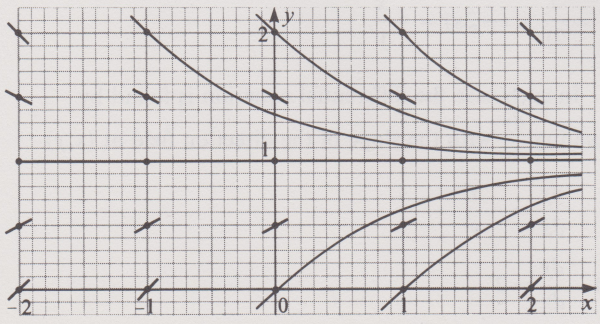
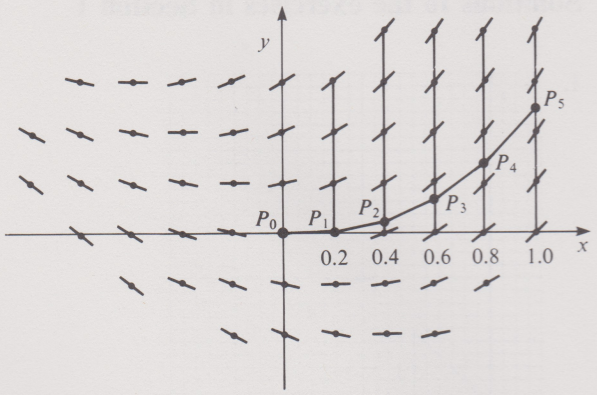



## Solutions to the exercises in Section 2

1.




The first line segment has slope 0 and ends at the point $P_1 = (0.2, 0)$. The next is drawn with a slope equal to that of the direction field at $P_1$ (the slope can be estimated from the diagram of the direction field, there is no need to calculate it) and continues until it reaches the point $P_2$ at which $x = 0.4$ (i.e. until it meets the vertical line $x = 0.4$). The next has slope equal to that of the direction field at $P_2$ and continues until it meets the vertical line $x = 0.6$, and so on.

2. The complete table is:

| $r$ | $x_r$ | $y_r$ | $m_r = x_r + y_r$ | $hm_r$ | $y_{r+1} = y_r + hm_r$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.2 | 0 | 0.2 | 0.04 | 0.04 |
| 2 | 0.4 | 0.04 | 0.44 | 0.088 | 0.128 |
| 3 | 0.6 | 0.128 | 0.728 | 0.1456 | 0.2736 |
| 4 | 0.8 | 0.2736 | 1.0736 | 0.21472 | 0.48832 |
| 5 | **1.0** | **0.48832** | | | |

The co-ordinates of $P_5$ are shown in bold type.

3. The only change to the column headings is that the formula for $m_r$ is now $y_r$ instead of $x_r + y_r$. Also, since the initial value $y(0)$ is now 1 instead of 0, the first entry in the $y_r$ column is now 1. The table is therefore:

| $r$ | $x_r$ | $y_r$ | $m_r = y_r$ | $hm_r$ | $y_{r+1} = y_r + hm_r$ |
|---|---|---|---|---|---|
| 0 | 0.0 | 1.0 | 1.0 | 0.2 | 1.2 |
| 1 | 0.2 | 1.2 | 1.2 | 0.24 | 1.44 |
| 2 | 0.4 | 1.44 | 1.44 | 0.288 | 1.728 |
| 3 | 0.6 | 1.728 | 1.728 | 0.3456 | 2.0736 |
| 4 | 0.8 | 2.0736 | 2.0736 | 0.41472 | 2.48832 |
| 5 | 1.0 | **2.48832** | | | |

The approximation to $y(1)$ is shown in bold type.

4. (i) The total change in $x$ is $2 - 1 = 1$, so with $n$ steps the step size is $1/n$.

(ii) Euler's formula for the differential equation can be written

$$y(x_{r+1}) \simeq y(x_r) + hx_r$$

so we have

(a) if $n = 1$ then $h = 1$ giving

$$y(2) \simeq y(1) + 1 \times 1 = 2 + 1 \times 1 = 3$$

so for 1 step $y(2) \simeq 3$,

(b) if $n = 2$ then $h = \frac{1}{2}$ giving

$y(1\frac{1}{2}) \simeq y(1) + \frac{1}{2} \times 1 = 2 + \frac{1}{2} \times 1 = 2\frac{1}{2}$

$y(2) \simeq y(1\frac{1}{2}) + \frac{1}{2} \times 1\frac{1}{2} \simeq 2\frac{1}{2} + \frac{1}{2} \times 1\frac{1}{2} = 3\frac{1}{4}$

so for 2 steps $y(2) \simeq 3.25$,

(c) if $n = 4$ then $h = \frac{1}{4}$ giving

$y(1\frac{1}{4}) \simeq 2 + \frac{1}{4} \times 1 = 2\frac{1}{4}$

$y(1\frac{1}{2}) \simeq 2\frac{1}{4} + \frac{1}{4} \times 1\frac{1}{4} = 2\frac{9}{16}$

$y(1\frac{3}{4}) \simeq 2\frac{9}{16} + \frac{1}{4} \times 1\frac{1}{2} = 2\frac{15}{16}$

$y(2) \simeq 2\frac{15}{16} + \frac{1}{4} \times 1\frac{3}{4} = 3\frac{3}{8}$

so for 4 steps $y(2) \simeq 3.375$.

(iii) The exact value of $y(2)$ is $\frac{1}{2}x^2 + \frac{3}{2}$ with $x = 2$, that is $y(2) = 3.5$.

For $n = 1$ the error is $3.5 - 3 = 0.5$.

For $n = 2$ the error is $3.5 - 3.25 = 0.25$.

For $n = 4$ the error is $3.5 - 3.375 = 0.125$.

(iv) It appears that the error is proportional to $h$, in fact the error seems to be equal to $\frac{1}{2}h$.

(v) For an accuracy of $10^{-4}$ we would need $\frac{1}{2}h = 10^{-4}$, that is $h = 2 \times 10^{-4}$. By (i) the number of steps would therefore be $1/h = 5000$.

## Solutions to the exercises in Section 3

**1.** (i) Integrating the function $6x^2$ gives the general solution

$y = 2x^3 + C$.

The required particular solution satisfies $y(1) = 5$, that is $2 \times 1^3 + C = 5$, which implies $C = 5 - 2 = 3$. The particular solution is therefore

$y = 2x^3 + 3$.

(ii) Integrating the function $e^{-2x}$ gives the general solution

$y = -\frac{1}{2}e^{-2x} + C$.

The required particular solution satisfies $y(0) = 2$, that is $-\frac{1}{2}e^0 + C = 2$, which implies $C = 2 + \frac{1}{2} = 2\frac{1}{2}$. The particular solution is therefore

$y = -\frac{1}{2}e^{-2x} + 2\frac{1}{2}$.

(iii) Integrating the function $a \sin bx$ gives the general solution

$$y = -\frac{a}{b}\cos bx + C.$$

The required particular solution satisfies $y(0) = 0$, that is $-\frac{a}{b}\cos(0) + C = 0$ which implies $C = \frac{a}{b}$. The particular solution is therefore

$$y = -\frac{a}{b}\cos bx + \frac{a}{b}.$$

**2.** We solve each equation by following the steps listed in Procedure 3.2.

(i) *Steps 1 to 3* tell us that we can integrate both sides of the equation to obtain

$$\int y^2\,dy = \int x^4\,dx + C.$$

*Step 4*: doing the integration gives

$\frac{1}{3}y^3 = \frac{1}{5}x^5 + C$.

*Step 5*: solving for $y$ we obtain the general solution

$y = \sqrt[3]{\frac{3}{5}x^5 + 3C}$.

*Step 6*: check:

$$\frac{dy}{dx} = \tfrac{1}{3}(\tfrac{3}{5}x^5 + 3C)^{-2/3} \times 3x^4 = \tfrac{1}{3}\frac{1}{y^2} \times 3x^4$$

hence $y^2\dfrac{dy}{dx} = x^4$.

(ii) *Steps 1 to 3* tell us that we can integrate both sides of the equation to obtain

$$\int e^y\,dy = \int \frac{1}{1+x^2}\,dx + C.$$

*Step 4*: doing the integrations we obtain

$e^y = \arctan x + C$

(arctan $x$ means the angle between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ whose tangent is $x$).

*Step 5*: the general solution is therefore

$y = \log_e(\arctan x + C)$.

*Step 6*: check:

$$\frac{dy}{dx} = \tfrac{1}{3}(\tfrac{3}{5}x^5 + 3C)^{-2/3} \times 3x^4 = \tfrac{1}{3}\frac{1}{y^2} \times 3x^4$$

hence $e^y\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

(iii) *Steps 1 to 3* tell us that we can integrate both sides of the equation to obtain

$$\int \frac{1}{1+y^2}\,dy = \int x\,dx + C.$$

*Step 4*: doing the integrations we obtain

$\arctan y = \frac{1}{2}x^2 + C$.

*Step 5*: the general solution is therefore

$y = \tan(\frac{1}{2}x^2 + C)$.

*Step 6*: check:

$$\frac{dy}{dx} = x\sec^2(\tfrac{1}{2}x^2 + C)$$

hence $$\frac{1}{1+y^2}\frac{dy}{dx} = \frac{x\sec^2(\frac{1}{2}x^2 + C)}{1 + \tan^2(\frac{1}{2}x^2 + C)}$$

$$= x \qquad (\text{since } \sec^2\theta = 1 + \tan^2\theta).$$

(iv) *Steps 1 to 3* tell us that we can integrate both sides of the equation to obtain

$$\int \frac{1}{y}\,dy = \int dx + C.$$

*Step 4*: doing the integrations we obtain

$\log_e y = x + C$ (since $y > 0$).

*Step 5*: the general solution is therefore

$y = \exp(x + C)$.

*Step 6*: check:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\exp(x + C)} \times \exp(x + C) = 1.$$

**3.** We solve each equation by first applying Procedure 3.3 and then following the steps listed in Procedure 3.2.

(i) After dividing by $e^y$ the equation becomes

$$e^{-y}\frac{dy}{dx} = e^x$$

which has the form described in *Step 1* of Procedure 3.2.

*Steps 2 and 3*: integrating and using the substitution rule gives

$$\int e^{-y}\,dy = \int e^{x}\,dx + C.$$

*Step 4*: doing the integrations we obtain

$$-e^{-y} = e^{x} + C.$$

*Step 5*: solving for $y$ gives the general solution

$$y = -\log_e(-e^{x} - C).$$

*Step 6*: check:

$$\frac{dy}{dx} = \left(\frac{-1}{-e^{x} - C}\right) \times (-e^{x}) = \left(\frac{-1}{e^{-y}}\right) \times (-e^{x}) = e^{x+y}.$$

(ii) The equation can be brought to the right form for Procedure 3.3 by dividing by $x$ giving

$$\frac{dy}{dx} = \frac{1}{x} \times y,$$

and we can bring this to the form described in *Step 1* of Procedure 3.2 by dividing by $y$ to obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}.$$

*Steps 2 and 3*: integrating and using the substitution rule gives

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx.$$

*Step 4*: doing the integrations we obtain

$$\log_e y = \log_e x + C \qquad (\text{since } x > 0 \text{ and } y > 0).$$

*Step 5*: solving for $y$ gives the general solution

$$y = \exp(\log_e x + C)$$
$$= xe^{C}.$$

*Step 6*: check:

$$x\frac{dy}{dx} = xe^{C} = y.$$

(iii) After multiplying by $y$ (or dividing by $1/y$) the equation becomes

$$y\frac{dy}{dx} = x$$

which has the form described in *Step 1* of Procedure 3.2.

*Steps 2 and 3*: integrating and using the substitution rule gives

$$\int y\,dy = \int x\,dx + C.$$

*Step 4*: doing the integrations we obtain

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C.$$

*Step 5*: solving for $y$ gives the general solution

$$y = \sqrt{x^2 + 2C}$$

*Step 6*: check: $y\dfrac{dy}{dx} = (x^2 + 2C)^{1/2}(x(x^2 + 2C)^{-1/2}) = x.$

**4.** (i) Dividing by $y$ and integrating both sides gives

$$\int \frac{1}{y}\,dy = \int (-A)\,dx + C.$$

hence $\log_e y = -Ax + C \qquad (\text{since } y > 0).$

Solving for $y$ gives the general solution

$$y = \exp(-Ax + C).$$

Check: $\dfrac{dy}{dx} = -A\exp(-Ax + C) = -Ay.$

(ii) Dividing by $y^2$ and integrating both sides gives

$$\int \frac{1}{y^2}\,dy = \int (-A)\,dx + C.$$

hence $-\dfrac{1}{y} = -Ax + C.$

Solving for $y$ gives the general solution

$$y = \frac{1}{Ax - C}.$$

Check: $\dfrac{dy}{dx} = -\dfrac{A}{(Ax - C)^2} = -Ay^2.$

(iii) Dividing by $B - y$ and integrating both sides gives

$$\int \frac{A}{B - y}\,dy = \int dx + C.$$

Doing the integrations (using the substitution $u = B - y$) we have

$$-\int \frac{A}{u}\,du = x + C$$

so $-A\log_e u = x + C \qquad (\text{since } u = B - y > 0)$

i.e. $-A\log_e(B - y) = x + C.$

The general solution is obtained by solving for $y$:

$$B - y = \exp\left(\frac{-x - C}{A}\right)$$

hence $y = B - e^{-x/A}e^{-C/A}.$

Check: $A\dfrac{dy}{dx} = e^{-C/A}e^{-x/A} = B - y.$

(iv) The same steps as in (iii) brings us to

$$-\int \frac{A}{u}\,du = x + C$$

where $u = B - y$. But since we now have $u < 0$ the integral gives

$$-A\log_e(-u) = x + C$$

i.e. $-A\log_e(-B + y) = x + C.$

The general solution is obtained by solving for $y$:

$$-B + y = \exp\left(\frac{-x - C}{A}\right)$$

i.e. $y = B + e^{-x/A}e^{-C/A}$

Check: $A\dfrac{dy}{dx} = -e^{-C/A}e^{-x/A} = B - y.$

(v) Dividing by $B^2 + y^2$ and integrating both sides gives

$$\int \frac{A}{B^2 + y^2}\,dy = \int dx + C.$$

Doing the integrations (using the substitution $y = B\tan u$ if necessary) we have

$$\frac{A}{B}\arctan\frac{y}{B} = x + C.$$

Solving for $y$ gives the general solution

$$y = B\tan\left(\frac{B(x + C)}{A}\right).$$

Check: $A\dfrac{dy}{dx} = B^2\sec^2\left(\dfrac{B(x + C)}{A}\right)$

$$= B^2\left(1 + \tan^2\left(\frac{B(x + C)}{A}\right)\right)$$

$$= B^2 + y^2.$$

**5.** We calculate each partial fraction following the steps listed in Procedure 3.4.

(i) *Step 1*: factorize the denominator to give

$$\frac{1}{x^2+x}=\frac{1}{x(x+1)}.$$

*Step 2*: assume $N_1$ and $N_2$ exist such that

$$\frac{1}{x(x+1)}=\frac{N_1}{x}+\frac{N_2}{x+1}.$$

*Step 3*: multiply both sides by $x(x+1)$ and collect terms to obtain

$$1=N_1(x+1)+N_2x$$
$$=(N_1+N_2)x+N_1$$

*Step 4*: it follows that

$$0=N_1+N_2$$
$$1=N_1$$

*Step 5*: this is a simultaneous equation with solution $N_1=1$, $N_2=-1$, the partial fraction expansion is therefore

$$\frac{1}{x^2+x}=\frac{1}{x}-\frac{1}{x+1}.$$

*Step 6*: as a check substitute $x=1$ into this expansion; both sides then equal $\frac{1}{2}$.

(ii) *Step 1*: the solutions of $x^2+x-2=0$ are $1$, $-2$. Hence

$$\frac{x+1}{x^2+x-2}=\frac{x+1}{(x-1)(x+2)}.$$

*Step 2*: assume $N_1$ and $N_2$ exist such that

$$\frac{x+1}{(x-1)(x+2)}=\frac{N_1}{x-1}+\frac{N_2}{x+2}.$$

*Step 3*: multiply both sides by $(x-1)(x+2)$ and collect terms to obtain

$$x+1=N_1(x+2)+N_2(x-1)$$
$$=(N_1+N_2)x+(2N_1-N_2).$$

*Step 4*: it follows that

$$1=N_1+N_2$$
$$1=2N_1-N_2.$$

*Step 5*: adding these equations together gives $3N_1=2$, that is $N_1=\frac{2}{3}$, substituting this into the first equation gives $N_2=\frac{1}{3}$, the partial fractions expansion is therefore

$$\frac{x+1}{x^2+x-2}=\frac{2}{3(x-1)}+\frac{1}{3(x+2)}.$$

*Step 6*: as check substitute $x=0$ into this expansion; both sides then equal $-\frac{1}{2}$.

(iii) *Step 1*: factorize the denominator to give

$$\frac{1}{B^2-y^2}=\frac{1}{(B-y)(B+y)}$$

*Step 2*: assume $N_1$ and $N_2$ exist such that

$$\frac{1}{(B-y)(B+y)}=\frac{N_1}{B-y}+\frac{N_2}{B+y}.$$

*Step 3*: multiply both sides by $(B-y)(B+y)$ and collect terms to obtain

$$1=N_1(B+y)+N_2(B-y)$$
$$=(N_1-N_2)y+(N_1B+N_2B).$$

*Step 4*: it follows that

$$0=N_1-N_2$$
$$1=N_1B+N_2B.$$

*Step 5*: adding $B$ times the first equation to the second gives $1=2N_1B$, that is $N_1=\dfrac{1}{2B}$, substituting this into the first equation gives $N_2=\dfrac{1}{2B}$; the partial fractions expansion is therefore

$$\frac{1}{B^2-y^2}=\frac{1}{2B(B-y)}+\frac{1}{2B(B+y)}.$$

*Step 6*: as a check substitute $y=0$ into this expansion; both sides then equal $1/B^2$.

(iv) The denominator of this expression is a cubic, but with minor modifications Procedure 3.4 can still be used. Factorize the denominator to give

$$\frac{1}{x(x^2-1)}=\frac{1}{x(x-1)(x+1)}$$

and assume $N_1$, $N_2$ and $N_3$ exist such that

$$\frac{1}{x(x-1)(x+1)}=\frac{N_1}{x}+\frac{N_2}{x-1}+\frac{N_3}{x+1}.$$

Multiply both sides by $x(x-1)(x+1)$ and collect terms to obtain

$$1=N_1(x^2-1)+N_2(x^2+x)+N_3(x^2-x)$$
$$=(N_1+N_2+N_3)x^2+(N_2-N_3)x-N_1$$

Since this must hold for all $x$ (except 0, 1 and $-1$) we have

$$0=N_1+N_2+N_3$$
$$0=N_2-N_3$$
$$1=-N_1.$$

From the third equation, $N_1=-1$, and from the second equation $N_2=N_3$. These two results, together with the first equation, imply that $0=-1+2N_2$, that is $N_2=N_3=\frac{1}{2}$. The partial fraction expansion is therefore

$$\frac{1}{x(x^2-1)}=-\frac{1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}.$$

As a check substitute $x=2$ into this expansion; both sides then equal $\frac{1}{6}$.

**6.** (i) After dividing both sides of

$$\frac{dy}{dx}=y^2+y$$

by $y^2+y$, we can integrate and use the substitution rule to obtain

$$\int\frac{1}{y^2+y}dy=\int dx+C.$$

Using the partial fraction expansion obtained in 5(i), this can be written

$$\int\left(\frac{1}{y}-\frac{1}{y+1}\right)dy=\int dx+C.$$

Doing the integrations, we obtain

$$\log_e y-\log_e(y+1)=x+C \qquad \text{(since } y>0\text{)}.$$

By the properties of logarithms, this can be written

$$\log_e\frac{y}{y+1}=x+C.$$

Taking exponentials of both sides and multiplying by $y+1$ gives

$$y=(y+1)\exp(x+C).$$

Solving for $y$ gives the general solution

$$y=\frac{\exp(x+C)}{1-\exp(x+C)}.$$

Check: $\dfrac{dy}{dx} = \dfrac{\exp(x+C)}{1-\exp(x+C)} + \dfrac{(\exp(x+C))^2}{(1-\exp(x+C))^2} = y + y^2.$

(ii) Using the partial fraction expansion found in 5(ii) the equation can be written

$$\frac{dy}{dx} = \frac{2}{3(x-1)} + \frac{1}{3(x+2)}.$$

Integrating both sides gives the general solution

$y = \frac{2}{3}\log_e(1-x) + \frac{1}{3}\log_e(x+2) + C$
(since $x+2>0$ and $1-x>0$).

Check:

$$\frac{dy}{dx} = \frac{2}{3(x-1)} + \frac{1}{3(x+2)}$$
$$= \frac{x+1}{x^2+x-2}.$$

(iii) After dividing both sides of the equation by $B^2 - y^2$, we can integrate both sides to obtain

$$\int \frac{A}{B^2-y^2}\,dy = \int dx + C.$$

Using the partial fraction expansion obtained in Exercise 5(iv) this can be written

$$A\int\left(\frac{1}{2B(B-y)} + \frac{1}{2B(B+y)}\right)dy = \int dx + C.$$

Carrying out the integrations we obtain

$$-\frac{A}{2B}\log_e(B-y) + \frac{A}{2B}\log_e(B+y) = x + C$$
(since $B-y>0$ and $B+y>0$).

This can be written

$$\log_e\frac{B+y}{B-y} = \frac{2B}{A}(x+C)$$

so $(B+y) = (B-y)\exp(2B(x+C)/A)$

hence $y = \dfrac{B\exp(2B(x+C)/A) - B}{1+\exp(2B(x+C)/A)}.$

Check: calculation of both $A\dfrac{dy}{dx}$ and $B^2 - y^2$ gives

$$\frac{4B^2\exp(2B(x+C)/A)}{(1+\exp(2B(x+C)/A))^2}.$$

## Solution to the exercise in Section 4

**1.** (i) This equation can be written in the form

$$p(x)\frac{dy}{dx} + \frac{dp(x)}{dx}y = q(x)$$

by choosing $p(x) = x$ and $q(x) = 1$, for then $\dfrac{dp(x)}{dx} = 1$. It follows that the equation can be written

$$\frac{d(xy)}{dx} = 1,$$

and can therefore be integrated directly to give

$$xy = x + C.$$

(ii) Here $p(x) = \sin x$, $\dfrac{dp(x)}{dx} = \cos x$, and $q(x) = \sin x$ and the given differential equation can be written

$$\frac{d((\sin x)y)}{dx} = \sin x.$$

It can therefore be integrated directly to give

$$(\sin x)y = -\cos x + C.$$

(iii) Here $p(x) = e^{3x}$, $\dfrac{dp(x)}{dx} = 3e^{3x}$ and $q(x) = e^{2x}$ and the given differential equation can be written

$$\frac{d(e^{3x}y)}{dx} = e^{2x}.$$

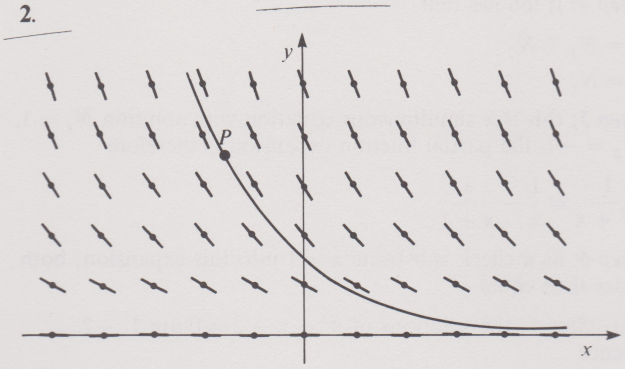
It can therefore be integrated directly to give

$$e^{3x}y = \tfrac{1}{2}e^{2x} + C.$$

## Solutions to the exercises in Section 5

**1.** (i) **A**, (methods B and C could also be used but A is the most straightforward), (ii) **B** or **C**, (iii) **B** or **C**, (iv) **C**, (v) **B**, (vi) **D**, (vii) **C**, (viii) **B**, (ix) **D**.

**2.**




**3.** (i) Euler's method gives the recurrence relation

$y_{r+1} = y_r + 0.1 \times (1 - x_r y_r)$

where $x_r = r \times 0.1$. The method starts from $x_0 = 0$, $y_0 = 1$.

(ii) the calculation is given in the following table

| $r$ | $x_r$ | $y_r$ | $x_r y_r$ | $1 - x_r y_r$ | $0.1 \times (1 - x_r y_r)$ | $y_{r+1}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0.1 | 1.1 |
| 1 | 0.1 | 1.1 | 0.11 | 0.89 | 0.089 | 1.189 |
| 2 | 0.2 | 1.189 | 0.2378 | 0.7622 | 0.07622 | 1.26522 |
| 3 | 0.3 | **1.26522** | | | | |

The Euler's approximation to $y(0.3)$ is shown in bold-face type.

**4.** (i) Since the error is roughly proportional to the step length, the error would be approximately halved.

(ii) The number of calculations would be approximately doubled.

**5.**

| Option | Response |
|---|---|
| 10 | 1 |
| 12 | 1 − X*Y |
| 22 | none |
| 36 | 0.001 |
| 33 | 0 |
| 34 | 1 |
| 35 | 300 |
| 40 | 100 |

**6.** Substituting the condition $y(4) = 8$ into the general solution gives $4C + 16 = 8$ therefore $C = -2$ and so the required particular solution is

$$y = -2x + x^2.$$

**7.** (i) The standard form of this equation is

$$\frac{dy}{dx} = -\frac{y^2}{x}.$$

Using the separation of variables method gives

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{1}{x} \qquad \text{(since } y > 0\text{)}$$

thus $$\int \frac{1}{y^2}\,dy = -\int \frac{1}{x}\,dx + C.$$

hence $$-\frac{1}{y} = -\log_e x + C \qquad \text{(since } x > 0\text{)}.$$

The general solution is therefore

$$y = \frac{1}{\log_e x - C}.$$

Check: $$\frac{dy}{dx} = \frac{-1}{x(\log_e x - C)^2} = -\frac{y^2}{x}.$$

(ii) The standard form of this equation is

$$\frac{dy}{dx} = x - \frac{2y}{x}.$$

Since this is linear we can use the integrating factor method. The integrating factor is

$$\exp\left(\int \frac{2}{x}\,dx\right) = \exp(2\log_e x) = x^2.$$

Multiplying our original equation by this factor gives

$$x^2\frac{dy}{dx} + 2xy = x^3$$

which can be written

$$\frac{d(x^2y)}{dx} = x^3.$$

Direct integration gives $x^2y = \frac{1}{4}x^4 + C$ and so the general solution is

$$y = \tfrac{1}{4}x^2 + \frac{C}{x^2}.$$

Check: $$\frac{dy}{dx} + \frac{2y}{x} = \frac{x}{2} - \frac{2C}{x^3} + \frac{x}{2} + \frac{2C}{x^3} = x.$$

(iii) This equation can be solved using separation of variables. Since we are only looking for positive solutions we can divide by $y + y^2$ and integrate to obtain

$$\int \frac{1}{y + y^2}\,dy = \int dx + C.$$

The right-hand side of this equation can be integrated to give $x + C$. The integration on the left-hand side can be carried out using partial fractions: let

$$\frac{1}{y(1+y)} = \frac{N_1}{y} + \frac{N_2}{1+y}$$

then

$$1 = N_1(1+y) + N_2y$$
$$= N_1 + (N_1 + N_2)y.$$

It follows that $N_1 = 1$ and $N_2 = -1$. Hence

$$\int \frac{1}{y+y^2}\,dy = \int\left(\frac{1}{y} - \frac{1}{1+y}\right)dy$$
$$= \log_e y - \log_e(1+y) \qquad \text{(since } y > 0\text{)}$$
$$= \log_e\left(\frac{y}{1+y}\right).$$

Equating the two sides of the equation gives

$$\log_e\left(\frac{y}{1+y}\right) = x + C$$

i.e. $$\frac{y}{1+y} = \exp(x + C).$$

The general solution can therefore be written

$$y = \frac{\exp(x+C)}{1 - \exp(x+C)}.$$

Check: $$\frac{dy}{dx} = \frac{\exp(x+C)}{1-\exp(x+C)} + \frac{(\exp(x+C))^2}{(1-\exp(x+C))^2} = y + y^2.$$

(iv) This equation is linear and can be solved using the integrating factor method. The integrating factor is

$$\exp\left(\int(-2)\,dx\right) = e^{-2x}.$$

Writing the equation in the form

$$\frac{dy}{dx} - 2y = -x$$

and multiplying by the integrating factor gives

$$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = -e^{-2x}x$$

which can be written

$$\frac{d(e^{-2x}y)}{dx} = -e^{-2x}x$$

so $$e^{-2x}y = -\int e^{-2x}x\,dx + C.$$

Integrating by parts we obtain

$$e^{-2x}y = \tfrac{1}{2}xe^{-2x} - \int \tfrac{1}{2}e^{-2x}\,dx + C$$
$$= \tfrac{1}{2}xe^{-2x} + \tfrac{1}{4}e^{-2x} + C.$$

The general solution is therefore

$$y = \tfrac{1}{2}x + \tfrac{1}{4} + Ce^{2x}.$$

Check: we have

$$\frac{dy}{dx} = \tfrac{1}{2} + 2Ce^{2x}$$

also

$$2y - x = x + \tfrac{1}{2} + 2Ce^{2x} - x = \tfrac{1}{2} + 2Ce^{2x}$$

and both sides agree.

**8.** The integration can be carried out using partial fractions. Since $x^2 + 3x + 2 = (x+1)(x+2)$ we can write

$$\frac{x+4}{x^2+3x+2} = \frac{N_1}{x+1} + \frac{N_2}{x+2}$$

therefore

$$x + 4 = N_1(x+2) + N_2(x+1)$$
$$= (N_1 + N_2)x + (2N_1 + N_2)$$

hence

$$N_1 + N_2 = 1$$
$$2N_1 + N_2 = 4$$

The solution of this simultaneous equation is $N_1 = 3$, $N_2 = -2$ and so

$$\frac{x+4}{x^2+3x+2} = \frac{3}{x+1} - \frac{2}{x+2}.$$

(Check: if $x = -4$ then both sides of this equation vanish.) We can now carry out the integration

$$\int \frac{x+4}{x^2+3x+2}\,dx = \int\left(\frac{3}{x+1} - \frac{2}{x+2}\right)dx$$
$$= 3\log_e(-(x+1)) - 2\log_e(-(x+2)) + C,$$

where we have used the fact that $x < -2$ implies $(x+1)$ and $(x+2)$ are negative.

## Solutions to home exercises on RECREL in Section 6

1. (i) The complete table is:

| Information | Option | Response |
|---|---|---|
| Formula for $m(x, y)$ | 12 | X + Y |
| Method (Euler) | 22 | none required |
| Initial value of $y$ | 34 | 0 |
| Initial value of $x$ | 33 | 0 |
| Number of steps | 35 | 5 |
| Step size | 36 | 0.2 |
| Type of print-out | 41 | none required |

(ii) The response to Option 36 becomes 0.002; Option 41 is replaced by Option 40, and the response to it is 50 because we only want every 50th calculated value (since $50 \times 0.002 = 0.1$, the interval between successive printed values of $x$).

2. (i) A suitable table is

| Information | Option | Response |
|---|---|---|
| Order of recurrence relation | 10 | 1 |
| Recurrence relation is | 11 | 1 − (R + 1)*U(R) |
| Method (forward recurrence) | 20 | none required |
| Initial value $u_0$ | 30 | 0.63212056 |
| Number of terms | 35 | 25 |
| Type of print-out | 41 | none required |

(ii) Part (i) can be left in the machine but the following new options must be inserted.

| Information | Option | Response |
|---|---|---|
| Method (backward recurrence) | 21 | none required |
| Value of $u_N$ | 32 | any number between 0 and 1 say |
| Number of terms | 35 | 35 |

*Note*: Inserting Option 21 automatically cancels Option 20 given in part (i), also, the response 35 to Option 35 automatically cancels the previous response of 25 to Option 35 given in part (i).

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